One-bit consensus of controllable linear multi-agent systems with communication noises

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***Abstract*— This paper addresses the one-bit consensus of controllable linear multi-agent systems (MASs) with communication noises. A consensus algorithm consisting of a communication protocol and a consensus controller is designed. The communication protocol introduces a linear compression encoding function to achieve a one-bit data rate, thereby saving communication costs. The consensus controller with a stabilization term and a consensus term is proposed to ensure the consensus of a potentially unstable but controllable MAS. Specifically, in the consensus term, we adopt an estimation method to overcome the information loss caused by one-bit communications and a decay step to attenuate the effect of communication noise. Two combined Lyapunov functions are constructed to overcome the difficulty arising from the coupling of the control and estimation. By establishing similar iterative structures of these two functions, this paper shows that the MAS can achieve consensus in the mean square sense at the rate of the reciprocal of the iteration number under the case with a connected fixed topology. Moreover, the theoretical results are generalized to the case with jointly connected Markovian switching topologies by establishing a certain equivalence relationship between the Markovian switching topologies and a fixed topology. Two simulation examples are given to validate the algorithm.**

***Index Terms*— consensus, one-bit data rate, communication noise, controllable linear MASs, Markovian switching topologies**

I. Introduction

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A. Background and motivation

OVER the past two decades, the consensus control of multi-agent systems (MASs) has been playing an increasingly important role in various fields, including engineering, communication, and biology. For example, in the engineering field, consensus control plays a crucial role in applications such as attitude alignment of satellites, rendezvous in space, and cooperative control of unmanned aerial vehicles [1]–[3]. In the communication field, it has been applied to problems like reputation consensus among mobile nodes [4] and load balancing in internet data centers [5]. In the biology field, consensus mechanisms are essential in understanding phenomena such as the aggregation behavior of animals [6] and the synchronous firing of biological oscillators [7].

With the increasing application of the consensus control across various fields, theoretical research on this topic has expanded significantly, such as [8]–[20]. Consensus controllers are typically formulated as the sum of the state differences between an agent and its neighbors, with the addition of a step coefficient. According to existing literature, the choice of the step coefficient usually depends on the MAS and significantly impacts the consensus rate. A constant step coefficient is often used to stabilize unstable systems [8]–[12]. Meanwhile, a decay step coefficient is commonly employed to attenuate the impact of noises, which is a widely adopted method in practice [13]–[20].

Due to the advantages of low communication costs and robustness, digital signals have become mainstream. It is known that data communication generally consumes significantly more energy and incurs higher costs compared to data processing [21]. These two factors make finite-bit data transmission between agents preferable and prevalent. Motivated by the advantages and challenges associated with finite-bit data, consensus control with finite-bit communications has attracted increasing attention in various fields.

B. Related literature

In fact, significant research has been conducted on the consensus control of MASs with finite-bit communications. Quantizer plays a crucial role in converting accurate communications into finite-bit communications in practical applications. Consequently, numerous studies have investigated consensus control using different quantizers, such as integer quantizers, logarithmic quantizers, uniform quantizers, binary-valued quantizers, and others. For example, Kashyap et al. in [22] and Chamie et al. in [23] considered the consensus control with integer quantized communications. Carli et al. in [24] introduced a logarithmic quantizer in the consensus control to improve the control performance. Li et al. in [25] proved that the MAS can achieve consensus with finite bits under a uniform quantizer in the noise-free case. Meng et al. in [26] extended [25] into a high-order system. Moreover, due to the significant reduction in communication costs offered by the binary-valued quantizer, the consensus control based on binary-valued communications has gained considerable research attention [27]–[30]. To be specific, Zhao et al. in [27] constructed a two-time-scale consensus algorithm and proved that the MAS can achieve mean square consensus with communication noises. Wang et al. in [28] proposed a consensus algorithm based on a recursive projection identification algorithm and obtained a mean square consensus rate, faster than that given by [27]. Wang et al. in [29] and An et al. in [30] extended the system of [28] to the high-order MAS under fixed and switching topologies, respectively. It is worth noting that the number of bits required for communication in the above consensus control depends not only on the choice of quantizer but also on the dimension of the agent's state. This implies that a one-bit data rate can be achieved only in first-order systems with binary-valued communications, as shown in [27]-[28], but the one-bit data rate cannot be achieved in [22]–[26], [29], [30].

In a communication network, the connectivity between agents significantly impacts the system's cooperation effectiveness. Most existing consensus research focuses on the fixed topology (such as [22]–[25], [27], [28]), while only a small portion addresses simpler cases with the switching topologies. Even for the switching topologies, certain restrictions remain. For example, the communication network is assumed to be periodically connected in [31] and [32], and modeled as an i.i.d. process in [30] and [33]. However, in practical applications, factors such as packet dropouts, environmental dynamics, link failures, and high-level scheduling commands lead to network topologies that switch with inherent correlations. Consequently, it becomes essential to model topology switching as a Markov process, which effectively captures the inherent correlations. Therefore, there is a need to study consensus control with finite-bit communications under both fixed topology and Markovian switching topologies.

C. Main contribution

In this paper, we consider the one-bit consensus of controllable linear MASs with communication noises for both fixed topology and Markovian switching topology cases. The main contributions of this paper are as follows:

- The system model studied in this paper is the most general linear system model of consensus control with finite-bit communications, requiring only controllability. Compared with previous studies of consensus with finite-bit communications [29]–[30], this paper removes the strict constraints of orthogonality and full row rank of the coefficient matrices, greatly expanding the applicability of the system model. Besides, this paper realizes the control of a high-order system with a first-order input, which simplifies the control process. To the best of the author's knowledge, it is the first consensus study on high-order systems under one-bit communications.
- A consensus algorithm consisting of a communication protocol and a consensus controller is proposed to achieve consensus with one-bit communications. In the communication protocol, a linear compression encoding function is introduced to compress state vectors into scalars, achieving a one-bit data rate and reducing communication costs compared to [26], [29], [30]. The consensus controller includes a stabilization term to ensure the stability of MASs and a consensus term with a decay step to attenuate the effect of stochastic communication noises. To overcome the information loss caused by the one-bit data rate, an estimation method is used in the consensus term to infer the neighbors' states from one-bit communications.
- The consensus properties of the proposed one-bit consensus algorithm are established under the connected fixed topology case. Two combined Lyapunov functions are constructed to overcome the difficulty arising from the coupling of the control and estimation. By establishing similar iterative structures of these two functions, this paper shows that the compressed states of MAS achieve consensus at a rate of $O(\frac{1}{t})$. Through establishing the consensus equivalence between the original and compressed states, it is shown that the MAS can achieve consensus in the mean square sense at a rate of $O(\frac{1}{t})$.
- The theoretical results are generalized to the case with jointly connected Markovian switching topologies by establishing a certain equivalence relationship between the Markovian switching topologies and a fixed topology. To be specific, the MAS can also achieve consensus in the mean square sense at a rate of $O(\frac{1}{t})$ under jointly connected Markovian switching topologies with appropriate step coefficients of estimation and controller. It is worth noting that the step coefficients depend on the switching probability of the Markovian switching topologies.

The remainder of this paper is organized as follows: Section II gives the preliminaries of basic concepts and graph theory and describes the consensus problem. Section III introduces the consensus algorithm. The main results of this paper are presented in Section IV, which includes the main convergence and consensus results. Section V generalizes the theoretical results in Section IV to Markovian switching topologies. Section VI gives two simulation examples for the fixed and switching topology cases. Section VII is the summary and prospect of this paper.

II. Preliminaries and problem formulation

In this section, we first give some basic concepts in matrix and graph theory, and subsequently formulate the system model and the consensus problems investigated in this paper.

A. Basic concept

Let $\mathbb{R}$ denote the set of real numbers, and $\mathbb{N}$ denote the set of natural numbers. We use $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times m}$ to denote n-dimensional column vector and $n \times m$-dimensional real matrix, respectively. Denote $\vec{0}_m = [0, \ldots, 0]^T \in \mathbb{R}^m$ and $\vec{1}_m = [1, \ldots, 1]^T \in \mathbb{R}^m$, where the notation T denotes the transpose operator. Denote $|a|$ as the absolute value of a scalar. Moreover, we denote $\|x\| = \|x\|_2$ and $\|A\| = \sqrt{\lambda_{\max}(A^T A)}$ as the Euclidean norm of vector and matrix, respectively, where $\lambda_{\max}(\cdot)$ denotes the largest eigenvalue of the matrix. Correspondingly, $\lambda_{\min}(\cdot)$ denotes the smallest eigenvalue of the matrix. For symmetric matrices $A \in \mathbb{R}^{m\times m}$ and $B \in \mathbb{R}^{m\times m}$, $A \geq B$ represents that $A - B$ is a positive semi-definite matrix. diag$\{\cdot\}$ denotes the block-diagonal matrix. And, for arbitrary matrices $A = [a_{ij}] \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$, the Kronecker product of A and B is defined as

$$A \otimes B \triangleq \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix} \in \mathbb{R}^{mp\times nq}.$$

In addition, the mathematical expectation is denoted as $E[\cdot]$. $\mathbb{D}$ denotes the one-step forward operator, i.e., let $x(k)$ be a sequence of variables, then $\mathbb{D}x(k) = x(k+1)$.

B. Graph theory

To describe the relation between agents, we introduce an undirected topology $G = (N_0, E)$, where $N_0 = \{1, \ldots, N\}$ is the set of agents, and $E \subseteq N_0 \times N_0$ is the ordered edges set of the topology G. Denote N_i as the neighbor set of the agent i in the topology G. Denote the adjacency matrix of the N agents as A_G, where each element of the matrix A_G satisfies $a_{ij} = a_{ji} = 1$ if $(i, j) \in E$, else $a_{ij} = 0$. Denote the degree matrix of G as D, where $D = \text{diag}\{d_1, d_2, \ldots, d_N\}$ and d_i is the degree of agent i. Denote the Laplace matrix of G as $L = D - A_G$. Denote $d_{\max} = \max_{1\leq i\leq N} d_i$ and $d = \sum_{i=1}^{N} d_i$. An undirected graph G is said to be connected if there exists a path between every pair of agents in G; otherwise, G is disconnected. If G is connected, then there exists an orthogonal matrix T_G such that $T_G^{-1} L T_G = \text{diag}\{\lambda_1, \ldots, \lambda_N\}$, where $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_N$ are the eigenvalues of the Laplacian matrix L.

C. Problem formulation

Consider the following MAS with N agents at time t:

$$x_i(t+1) = Ax_i(t) + Bu_i(t), \quad i \in N_0, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^n$ are constant matrices, $x_i(t) = [x_{i1}(t), \ldots, x_{in}(t)]^T \in \mathbb{R}^n$ is the state of the agent i at time t, and $u_i(t) \in \mathbb{R}$ is the control input of the agent i at time t.

Agent i receives one-bit information $s_{ij}(t)$ affected by communication noise from its neighbor j:

$$s_{ij}(t) = \mathbb{1}_{\{g(x_j(t))+d_{ij}(t)\leq c_{ij}\}}, \tag{2}$$

where the agent j is the neighbor of the agent i at time t, $g(\cdot) : \mathbb{R}^n \to \mathbb{R}$ is a compression encoding function to be designed, $d_{ij}(t) \in \mathbb{R}$ is the communicating noise, $c_{ij} \in \mathbb{R}$ is the threshold value, $s_{ij}(t)$ is the one-bit information that the agent i collects from its neighbor j, $\mathbb{1}_{\{a\leq c\}}$ is the indicator function defined as:

$$\mathbb{1}_{\{a\leq c\}} = \begin{cases} 1, & a \leq c, \\ 0, & a > c. \end{cases}$$

Remark 1: The compression encoding function $g(\cdot)$ is a common tool to save communication costs in the communication field [34]–[36], as it provides savings of scarce network resources such as communication bandwidth, transmit/processing power, and storage. In contrast with [26], [29], [30], the compression encoding function $g(\cdot)$ maps n-dimensional vectors to scalars to realize one-bit data rate communication, but agents in [26], [29], [30] need to transmit finite-bit data depending on the dimension of states.

To proceed with our analysis, we introduce two assumptions about the system model and the communication noises.

Assumption 1: The linear system (A, B) is controllable.

Assumption 2: The noises $\{d_{ij}(t), i, j \in N_0, t \in \mathbb{N}\}$ are independent and identically distributed as $N(0, \delta^2)$ for indices i, j and time t, with known distribution function $F(\cdot)$ and density function $f(\cdot)$.

Assumption 3: The topology graph G is connected.

Remark 2: Compared with the existing consensus works based on finite-bit communications, the condition of the system model in this paper is the weakest, requiring only controllability, as stated in Assumption 1. To be specific, in [26], besides the requirement of controllability for (A, B), there were restrictions on the eigenvalues of the coefficient matrix A. In [29], orthogonality constraints on the coefficient matrices were required. In [30], the system model is assumed to be neutrally stable.

Definition 1: ([37, Definition 3] **Mean square consensus**). The agents' states $x_i(t)$ are said to achieve mean square consensus if $E[\|x_i(t)\|^2] < \infty$, $t \geq 0$, $i \in N_0$, and there exists a random variable x^* such that $\lim_{t\to\infty} E[\|x_i(t) - x^*\|^2] = 0$ for all $i \in N_0$.

Problem: The goal of this paper is to design a consensus algorithm comprising a controller $u_i(t)$ and a communication mechanism $g(\cdot)$ based on one-bit communications $s_{ij}(t)$ to achieve consensus of the controllable MAS (1)-(2).

III. Algorithm design

This section focuses on designing a consensus algorithm that enables linear systems to reach a consensus with communication noises and a one-bit data rate constraint.

To simplify the design of the algorithm, this paper considers the MAS (1)-(2) in the Brunovsky canonical form, where

$$A = \tilde{A} \triangleq \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \end{bmatrix} \in \mathbb{R}^{n\times n}, B = \tilde{B} \triangleq \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^n. \tag{3}$$

It can be seen that for any controllable system (1), there exists a nonsingular matrix P that can transform (1) into this Brunovsky canonical form [38], i.e., $PAP^{-1} = \tilde{A}$ and $PB = \tilde{B}$. Let $\tilde{x}_i(t) = Px_i(t)$. Then, (1) is transformed into $\tilde{x}_i(t+1) = \tilde{A}\tilde{x}_i(t) + \tilde{B}u_i(t)$, which is equivalent to (1)-(3).

Next, we introduce the design idea of the consensus algorithm. To achieve one-bit consensus for general controllable systems, this paper proposes an integrated algorithm consisting of a communication protocol and a consensus controller. On the one hand, to enable one-bit communication for high-order systems, a linear compression-based communication protocol is designed to efficiently compress each agent's high-dimensional state vector into a scalar value. On the other hand, to address the state unknown caused by one-bit communication, a consensus controller incorporating both estimation and control is developed.

Based on the above idea, we propose a one-bit consensus algorithm involving both a communication protocol and a consensus controller in Algorithm 1.

Algorithm 1 One-bit consensus algorithm

Initiation: Denote the integer $t_0(>0)$ as the initial time. $x_i(t_0+1)=x_i^0$ is the initial state of the agent i, $\hat{z}_{ij}(t_0)=\hat{z}_{ij}^0$ is the initial estimate of the agent j estimated by the agent i. Then, for $t\geq t_0+1$, the algorithm is as follows.

Step 1: Communication protocol:

Denote the compression encoding function

$$g(x_j(t))=K_2x_j(t),$$

where $K_2=[b_1,b_2,\ldots,b_{n-1},1]\in\mathbb{R}^{1\times n}$, where b_1, b_2, $\ldots$, b_{n-1} are the compression coefficients to be designed. Then, the one-bit communcation (2) is $s_{ij}(t)=\mathbb{1}_{\{K_2x_j(t)+d_{ij}(t)\leq c_{ij}\}}$.

Step 2: Consensus controller:

Step 2.1 Estimation: each agent i estimates the compressed state $K_2x_j(t)$ of its neighbor agent j at time t by

$$\hat{z}_{ij}(t)=\Pi_M\Big\{\hat{z}_{ij}(t-1)+\frac{\beta}{t}\Big(F\big(c_{ij}-\hat{z}_{ij}(t-1)\big)-s_{ij}(t)\Big)\Big\}, \tag{4}$$

where $j\in N_i$, β is the step coefficient for estimation updating, $F(\cdot)$ is the distribution function of noise $d_{ij}(t)$, $\Pi_M(\cdot)$ is a projection mapping defined as

$$\Pi_M(\zeta)=\underset{|\xi|\leq M}{\arg\min}\,|\zeta-\xi|,\forall\zeta\in\mathbb{R}, \tag{5}$$

where $M=\max_{i\in N_0,j\in N_i}\{|K_2x_i^0|,|\hat{z}_{ij}^0|\}$.

Step 2.2 Control: based on these estimates, each agent i designs its control by

$$u_i(t)=K_1x_i(t)+\frac{\gamma d_i}{t+1}\sum_{j\in N_i}\big(\hat{z}_{ij}(t)-K_2x_i(t)\big), \tag{6}$$

where $K_1=[-a_1+b_1,-a_2+b_2-b_1,\ldots,-a_{n-1}+b_{n-1}-b_{n-2},-a_n-b_{n-1}+1]\in\mathbb{R}^{1\times n}$ and γ is the step coefficient of the control that needs to be designed.

Remark 3: For the communication protocol, the compression encoding function saves communication costs, but it complicates the recovery of the original states $x_j(t)$ from the compressed states $K_2x_j(t)$. Noting that the linear compression encoding function $K_2x_j(t)$ is irreversible, the key issue to be explored is how to design the linear compression coefficient K_2 to ensure the equivalence between the consensus of original and compressed states, thereby achieving consensus of the original states through the consensus of the compressed states. This design problem of K_2 and its theoretical justification will be addressed and rigorously proved in Section IV-B.

Remark 4: The estimation-control type consensus controller, consisting of the estimation part (4) and the control part (6), is designed to handle the unknown state caused by one-bit communication. Specifically, the estimation part (4) is used to reconstruct the neighbors' compressed states $K_2x_j(t)$ based on the received one-bit data $s_{ij}(t)$, and the control part (6) is designed to achieve consensus for controllable MASs under communication noise. The control part includes a stabilization term to address system instability and a consensus term with a decaying step to attenuate the effect of stochastic communication noise. Besides, the upper bound M in the estimation step only requires M to exceed $\max_{i\in N_0,\,j\in N_i}\big\{|K_2x_i^0|,\,|\hat{z}_{ij}^0|\big\}$, without requiring exact equality.

Remark 5: Algorithm 1 is applicable to not only the Brunovsky canonical form but also general controllable systems, which requires only a transformation in the control gain K_1 and compression coefficient K_2. Specifically, K_1P and K_2P are used as the control gain and the compression coefficient for general controllable systems, where P is the transformation matrix for the Brunovsky canonical form. Besides, control gains K_1 and K_2 designed in Algorithm 1 satisfy

$$K_2(A+BK_1)=K_2\quad\text{and}\quad K_2B=1.$$

Remark 6: The proposed algorithm trades a small amount of local computation and memory for a substantial reduction in overall communication costs, compared to previous works (e.g., [8]–[20], [22]–[26], [29], [30]). Specifically, the computational and storage complexities required by the proposed estimation algorithm are both low, with the former being approximately 8-9 flops per neighbor per time step and the latter requiring only one scalar per neighbor. Meanwhile, the communication cost is significantly reduced to just one bit per unit time for each neighbor.

IV. Main reslults

In this section, we demonstrate that all agents can reach consensus and provide the corresponding consensus rate.

At first, we analyze the consensus property of the compressed states and the convergence property of their estimates. Then, the equivalence between the consensus of the compressed and original states is established. Finally, the consensus of the original states and the corresponding consensus rate are obtained.

A. Properties of compressed states and their estimates

For the convenience of the subsequent analysis, we rewrite the estimation and update formulas in vector form based on the fixed topology G.

Firstly, define $x(t)=[x_1^T(t),x_2^T(t),\ldots,x_N^T(t)]^T\in\mathbb{R}^{nN}$. Let $\hat{z}(t)=[\ldots,\hat{z}_{ij}(t),\ldots]^T\in\mathbb{R}^d$ denote the stacking of all $\hat{z}_{ij}(t)$, arranged in lexicographic order of $(i,j)\in E$, i.e., sorted first by the index i and then by j when i is the same. The same ordering is used for $s(t)=[\ldots,s_{ij}(t),\ldots]^T\in\mathbb{R}^d$ and $C=[\ldots,c_{ij}(t),\ldots]^T\in\mathbb{R}^d$.

Then, we construct two matrices to establish the relation between agents' compressed states and their estimates.

Q is designed to select the true compressed state corresponding to its estimate. Define $Q = [\dots, Q_{ij}, \dots]^T \in \mathbb{R}^{d\times N}$ in lexicographic order of $(i, j) \in E$, where $Q_{ij} = \vec{e}_j \in \mathbb{R}^N$, $\vec{e}_j$ is the j-th canonical vector with a 1 in the j-th position and 0 elsewhere.

W is designed to select the neighbor set of each agent. Define $W = [\dots, W_{ij}, \dots] \in \mathbb{R}^{N\times d}$ in lexicographic order of $(i, j) \in E$, where $W_{ij} = \vec{e}_i \in \mathbb{R}^N$.

Based on the above matrices, the vector forms of estimation and update are given as follows:

1.Estimation:

$$\hat{z}(t) = \mathbf{\Pi}_{\boldsymbol{M}}\Big\{\hat{z}(t-1) + \frac{\beta}{t}\big(\mathcal{F}(C - \hat{z}(t-1)) - s(t)\big)\Big\},$$

where $\mathbf{\Pi}_{\boldsymbol{M}}(z) = [\Pi_M(z_1), \dots, \Pi_M(z_d)]^T$, $\mathcal{F}(z) = [F(z_1), \dots, F(z_d)]^T$, for any $z = [z_1, z_2, \dots, z_d]^T \in \mathbb{R}^d$.

2.Update:

$$x(t) = \Big(I_N \otimes (A + BK_1) - \frac{\gamma}{t} L \otimes BK_2\Big)x(t-1) + \frac{\gamma}{t}(W \otimes B)\hat{\varepsilon}(t),$$

where $\hat{\varepsilon}(t) = \hat{z}(t) - (Q \otimes K_2)x(t)$ is the estimation error of the compressed state $(I_N \otimes K_2)x(t)$.

As a preparatory step for analyzing consensus and estimation convergence, we provide the following lemma on the boundedness of the compressed states and their estimates.

Lemma 1: Under Assumptions 1-3 and Algorithm 1, the compressed states $K_2x_i(t)$ and their estimates $\hat{z}_{ij}(t)$ are all bounded for all $t \geq t_0 + 1$, i.e.,

$$|K_2x_i(t)| \leq M \quad \text{and} \quad |\hat{z}_{ij}(t)| \leq M,$$

where M is defined as (4), $i \in N_0$, $j \in N_i$.

Proof: First, due to the definition of M, we can get $|K_2x_i^0| \leq M, |z_{ij}^0| \leq M$. By (4) and (5), we have $|\hat{z}_{ij}(t)| \leq M$ for $t \geq t_0 + 1$.

Next, we prove the boundedness of the compressed state $K_2x_i(t)$ by mathematical induction.

Base case: For $t = t_0$, there is $|K_2x_i^0| \leq M$.

Inductive step: We show that if $|K_2x_i(t)| \leq M$ holds for an arbitrary $t \geq t_0$, then it holds for $t + 1$.

Assume that $|K_2x_i(t)| \leq M$, then by the system model (1), control (6), and properties of K_1, K_2 in Remark 5, we have

$$\begin{aligned}
&|K_2x_i(t+1)| \\
=&\Big|K_2(A + BK_1)x_i(t) + \frac{\gamma K_2 B}{t+1}\sum_{j\in N_i}\big(\hat{z}_{ij}(t) - K_2x_i(t)\big)\Big| \\
=&\Big|K_2x_i(t) + \frac{\gamma}{t+1}\sum_{j\in N_i}\left(\hat{z}_{ij}(t) - K_2x_i(t)\right)\Big|. \\
=&\Big|\left(1 - \frac{d_i\gamma}{t+1}\right)K_2x_i(t) + \frac{\gamma}{t+1}\sum_{j\in N_i}\hat{z}_{ij}(t)\Big|. \\
\leq&\Big|1 - \frac{\gamma d_i}{t+1}\Big|M + \frac{\gamma d_i}{t+1}M
\end{aligned}$$

Without loss of generality, we assume that $t_0 \geq \gamma d_{\max} - 1$. Then, we have $\frac{\gamma d_i}{t+1} < \frac{\gamma d_{\max}}{t_0+1} \leq 1$. Furthermore, by $1 - \frac{\gamma d_i}{t+1} + \frac{\gamma d_i}{t+1} = 1$, we can get

$$|K_2x_i(t+1)| \leq \Big|1 - \frac{\gamma d_i}{t+1}\Big|M + \frac{\gamma d_i}{t+1}M = M.$$

Thus, by mathematical induction, we have $|K_2x_i(t)| \leq M$ for all $t \geq t_0 + 1$. The lemma is proved. ■

Remark 7: The assumption $t_0 \geq \gamma d_{\max} - 1$ used in the proof is made without loss of generality, as it can always be satisfied by adjusting the step size $\frac{\gamma}{t+1}$ of the consensus term in (6). Specifically, when $t_0 < \gamma d_{\max} - 1$, we can simply modify the step from $\frac{\gamma}{t+1}$ to $\frac{\gamma}{t+t_\gamma}$, where $t_\gamma = \gamma d_{\max}$. This modification ensures that $\frac{\gamma}{t+t_\gamma} \leq \frac{1}{d_{\max}}$ holds for all $t \geq t_0$, which is sufficient for the boundedness analysis. Since t_γ is a constant independent of the initial time t_0, this change has no impact on the consensus properties.

Next, due to the coupling relationship between the control and estimation process, we introduce two Lyapunov functions, $V(t)$ and $R(t)$, to jointly analyze the consensus of the compressed states and the convergence of their estimates. These functions are defined as follows:

$$\begin{aligned}
V(t) &= E[\|(T_G^{-1} \otimes K_2)\delta(t)\|^2], \\
R(t) &= E[\|\hat{\varepsilon}(t)\|^2],
\end{aligned}$$

where $\delta(t) = (J_N \otimes I_n)x(t)$ and $J_N = I_N - \frac{1}{N}\vec{1}_N\vec{1}_N^T$, T_G is defined in Section II-B. Then, the following two lemmas show the coupled expressions of the two Lyapunov functions.

Lemma 2: Under Assumptions 1-3, $V(t)$ satisfies

$$V(t) \leq \Big(1 - \frac{\gamma\lambda_2}{t}\Big)V(t-1) + \frac{\gamma\lambda_G}{t\lambda_2}R(t-1) + O\Big(\frac{1}{t^2}\Big),$$

where $\lambda_G = \|J_N W\|^2$.

Proof: See Appendix I-A. ■

Lemma 3: Under Assumptions 1-3, $R(t)$ satisfies

$$R(t) \leq \Big(1 - \frac{2\beta f_M - \gamma\alpha}{t}\Big)R(t-1) + \frac{\gamma\lambda_G}{t\lambda_2}V(t-1) + O\Big(\frac{1}{t^2}\Big),$$

where $\alpha = 2\sqrt{\lambda_{QW}} + \lambda_{QL}\lambda_2/\lambda_G$, $f_M = \min_{i,j\in N_0}\{f(|c_{ij}| + M)\}$, $\lambda_{QW} = \|QW\|^2$, and $\lambda_{QL} = \|QLT_G\|^2$.

Proof: See Appendix I-B. ■

To establish the convergence properties of these two coupled functions $V(t)$ and $R(t)$, denote a new function $Z(t) = (V(t), R(t))^T$. Then, under Assumptions 1-3, we have

$$\|Z(t)\| \leq \|(I - \frac{1}{t}U)Z(t-1)\| + O\Big(\frac{1}{t^2}\Big),$$

where $U = \begin{bmatrix} u_1 & u_2 \\ u_2 & u_4 \end{bmatrix}$, $u_1 = \gamma\lambda_2$, $u_2 = \gamma\lambda_G/\lambda_2$, $u_4 = 2\beta f_M - \gamma\alpha$, α is the same as in Lemma 3. Since $0 \leq V(t) \leq \|Z(t)\|$ and $0 \leq R(t) \leq \|Z(t)\|$, the analysis of $V(t)$ and $R(t)$ can be transformed into analyzing the convergence of $Z(t)$.

Then, the following theorem is established to show the consensus properties of the compressed states.

Theorem 1: Under Assumptions 1-3, the compressed states $K_2x_j(t)$ and their estimates $\hat{z}_{ij}(t)$ satisfy:

i) If $\beta > \frac{1}{2f_M}(\frac{\gamma\lambda_G^2}{\lambda_2^3} + \gamma\alpha)$, the compressed states reach consensus and their estimates converge to the real compressed states, i.e., for $i \in N_0$, $j \in N_i$,

$$\lim_{t\to\infty} E[\|K_2x_i(t) - K_2\bar{x}(t)\|^2] = 0,$$

$$\lim_{t\to\infty} E[\|\hat{z}_{ij}(t) - K_2x_j(t)\|^2] = 0;$$

ii) If $\beta > \frac{1}{2f_M}(\frac{\gamma^2\lambda_G^2}{\lambda_2^2(\gamma\lambda_2-1)} + \gamma\alpha + 1)$ and $\gamma > \frac{1}{\lambda_2}$, the compressed states reach consensus at the rate of $O(\frac{1}{t})$, and their estimates converge to the real compressed states at the rate of $O(\frac{1}{t})$, i.e., for $i \in N_0$, $j \in N_i$,

$$E[\|K_2x_i(t) - K_2\bar{x}(t)\|^2] = O\Big(\frac{1}{t}\Big),$$

$$E[\|\hat{z}_{ij}(t) - K_2x_j(t)\|^2] = O\Big(\frac{1}{t}\Big),$$

where $\bar{x}(t) = \frac{1}{N}\sum_{i=1}^{N} x_i(t)$.

Proof: i) By [33, Theorem 1], we have

$$\|Z(t)\| = \begin{cases} O\left(\frac{1}{t^{\lambda_{\min}(U)}}\right), & \lambda_{\min}(U) < 1; \\ O\left(\frac{\ln t}{t}\right), & \lambda_{\min}(U) = 1; \\ O\left(\frac{1}{t}\right), & \lambda_{\min}(U) > 1. \end{cases} \quad (7)$$

Let $|\lambda I_2 - U| = \lambda^2 - (u_1 - u_4)^2 + 4u_2^2 = 0$. Then, we have

$$\lambda_{\min}(U) = \frac{1}{2}\Big(u_1 + u_4 - \sqrt{(u_1 + u_4)^2 - 4(u_1u_4 - u_2^2)}\Big).$$

If $\beta > \frac{1}{2f_M}(\frac{\gamma\lambda_G^2}{\lambda_2^3} + \gamma\alpha)$, then $u_1u_4 > u_2^2$. Since $u_1 > 0$ and $u_1u_4 > u_2^2$, we have $\lambda_{\min}(U) > 0$. Then, we can obatin that $\lim_{t\to\infty}\|Z(t)\| = 0$ by Equation (7). Hence, due to $0 \le V(t), R(t) \le \|Z(t)\|$, it is clearly that

$$\lim_{t\to\infty} V(t) = 0, \quad \lim_{t\to\infty} R(t) = 0,$$

which is equivalent to $\lim_{t\to\infty} E[\|(T_G^{-1} \otimes K_2)\delta(t)\|^2] = 0$ and $\lim_{t\to\infty} E[\|\hat{\varepsilon}(t)\|^2] = 0$.

Subsequently, there is $\lim_{t\to\infty} E[\|I_N \otimes K_2)\delta(t)\|^2] = 0$, which implies that $\lim_{t\to\infty} E[\|K_2\delta_i(t)\|^2] = 0$ for $i \in N_0$. Therefore, we have that for $i \in N_0$, $j \in N_i$,

$$\lim_{t\to\infty} E[\|K_2x_i(t) - K_2\bar{x}(t)\|^2] = 0,$$

$$\lim_{t\to\infty} E[\|\hat{z}_{ij}(t) - K_2x_j(t)\|^2] = 0.$$

ii) Similarly, if $\beta > \frac{1}{2f_M}(\frac{\gamma^2\lambda_G^2}{\lambda_2^2(\gamma\lambda_2-1)} + \gamma\alpha + 1)$, then

$$\frac{u_2^2}{u_1 - 1} + 1 < u_4.$$

If $\gamma > \frac{1}{\lambda_2}$, we have $u_1 = \gamma\lambda_2 > 1$, then

$$u_2^2 + u_1 - 1 < u_4(u_1 - 1).$$

Subsequently, since $(u_1+u_4)^2 - 4(u_1u_4 - u_2^2) - (u_1+u_4-2)^2 = 4(u_2^2 + u_1 - 1 - u_4(u_1 - 1)) < 0$, we have

$$u_1 + u_4 - \sqrt{(u_1 + u_4)^2 - 4(u_1u_4 - u_2^2)} > 2,$$

then $\lambda_{\min}(U) > 1$.

By Equation (7), we have $\|Z(t)\| = O(\frac{1}{t})$, i.e.,

$$V(t) = O\left(\frac{1}{t}\right), \quad R(t) = O\left(\frac{1}{t}\right).$$

Similarly to the proof of Part i), we have

$$E[\|K_2x_i(t) - K_2\bar{x}(t)\|^2] = O\Big(\frac{1}{t}\Big),$$

$$E[\|\hat{z}_{ij}(t) - K_2x_j(t)\|^2] = O\Big(\frac{1}{t}\Big).$$

where $i \in N_0$, $j \in N_i$. ■

B. Consensus of the original states

In order to establish the equivalence of the consensus of the agent states before and after compression, we need the compression coefficients to satisfy the following condition.

Assumption 4: The compression coefficients $b_1, b_2, \ldots, b_{n-1}$ satisfy that: All the roots $r_1, r_2, \ldots, r_{n-1}$ of $s^{n-1} + b_{n-1}s^{n-2} + \cdots + b_2s + b_1 = 0$ are inside the unit circle.

Remark 8: Assumption 4 provides a method for designing the compression coefficient K_2 that guarantees the equivalence between the consensus of the compressed states $K_2x_i(t)$ and that of the original states $x_i(t)$. Specifically, Assumption 4 essentially serves as the minimum-phase condition of $B(s) = s^{n-1} + b_{n-1}s^{n-2} + \cdots + b_2s + b_1$. This ensures that, for the dynamic equation $\eta(t) = B(s)\xi(t)$ with $B(s)$ as its coefficient, the convergence of $\eta(t)$ guarantees the convergence of $\xi(t)$, where s is the one-step forward operator. Therefore, once it can be shown that the compressed state and the original state satisfy the above dynamic equation, the compression-coefficient condition given in Assumption 4 ensures the equivalence of consensus between the compressed state and the original state.

The following two lemmas respectively present the dynamic relationship between the compressed state and the original state, and the equivalence of consensus between the variables in a dynamic equation whose coefficients satisfy the condition in Assumption 4.

Lemma 4: The linear systems (1) with Brunovsky canonical form (3) satisfies

$$\mathbb{D}^{n-1}x_{in}(t) + b_{n-1}\mathbb{D}^{n-2}x_{in}(t) + \cdots + b_1x_{in}(t) = \mathbb{D}^{n-1}K_2x_i(t),$$

where $x_i(t) = [x_{i1}(t), \ldots, x_{in}(t)]^T \in \mathbb{R}^n$ and $i \in N_0$.

Proof: See Appendix II-A. ■

Lemma 5: Consider the stochastic process $\xi(t) \in \mathbb{R}$ that satisfies the following stochastic difference equation:

$$\mathbb{D}^{n-1}\xi(t) + b_{n-1}\mathbb{D}^{n-2}\xi(t) + \cdots + b_2\mathbb{D}\xi(t) + b_1\xi(t) = \eta(t),$$

where $\eta(t) \in \mathbb{R}$ is a stochastic process which converges to a finite-second-moment random variable η^* in the mean square. Under Assumption 4,

i) $\lim_{k\to\infty} E[|\xi(t) - \xi^*|^2] = 0$;

ii) If $E[|\eta(t) - \eta^*|^2] = O(\frac{1}{t})$, then $E[|\xi(t) - \xi^*|^2] = O(\frac{1}{t})$,

where $\xi^* = \frac{1}{\prod_{j=1}^{n-1}(1-r_j)}\eta^*$ and $r_1, \ldots, r_{n-1}$ are defined as Assumption 4.

Proof: See Appendix II-B. ■

Lemmas 4-5 demonstrate that the consensus of original states is equivalent to that of the compressed states. Specifically, Lemma 4 shows that $x_{in}(t)$ and $K_2x_i(t)$ satisfy the difference equation in Lemma 5 for each agent i. Since $x_i(t) - \bar{x}(t)$ is a linear combination of $x_1(t), \ldots, x_N(t)$, it can be seen that $x_{in}(t) - \bar{x}_n(t)$ and $K_2(x_i(t) - \bar{x}(t))$ still satisfy the difference equation in Lemma 5, where $\bar{x}(t) = [\bar{x}_1(t), \ldots, \bar{x}_n(t)]^T \in \mathbb{R}^n$. Then, the equivalence between the consensus of the original states $x_{in}(t) - \bar{x}_n(t)$ and compressed states $K_2(x_i(t) - \bar{x}(t))$ is given in Lemma 5.

Using the consensus equivalence between the original states $x_j(t)$ and the compressed states $K_2x_j(t)$, the consensus of

MAS can be established based on the results of the compressed states in Theorem 1 as follows.

Theorem 2: Under Assumptions 1-4, the following results are obtained:

i) The MAS (1)-(3) reaches consensus, i.e.,

$$\lim_{t\to\infty} E[\|x_i(t)-\bar{x}(t)\|^2]=0,$$

providing with $\beta>\frac{1}{2f_M}(\frac{\gamma\lambda_G^2}{\lambda_2^3}+\gamma\alpha)$;

ii) The MAS (1)-(3) reaches consensus at the rate of $O(\frac{1}{t})$, i.e.,

$$E[\|x_i(t)-\bar{x}(t)\|^2]=O\left(\frac{1}{t}\right),$$

if $\beta>\frac{1}{2f_M}(\frac{\gamma^2\lambda_G^2}{\lambda_2^2(\gamma\lambda_2-1)}+\gamma\alpha+1)$ and $\gamma>\frac{1}{\lambda_2}$.

Proof: i) By Lemmas 4-5, we have

$$\begin{aligned}&\mathbb{D}^{n-1}\big(x_{in}(t)-\bar{x}_n(t)\big)+b_{n-1}\mathbb{D}^{n-2}\big(x_{in}(t)-\bar{x}_n(t)\big)\\&+\cdots+b_1\big(x_{in}(t)-\bar{x}_n(t)\big)\\=&\mathbb{D}^{n-1}K_2\big(x_i(t)-\bar{x}(t)\big).\end{aligned}\tag{8}$$

By Theorem 1, when $\beta>\frac{1}{2f_M}(\frac{\gamma\lambda_G^2}{\lambda_2^3}+\gamma\alpha)$, there is $\lim_{t\to\infty}E[\|K_2x_i(t)-K_2\bar{x}(t)\|^2]=0$ for $i\in N_0$.

Then, using Lemma 5 and (8), we have

$$\lim_{t\to\infty}E[\big(x_{in}(t)-\bar{x}_n(t)\big)^2]=0.$$

The proof of Lemma 4 shows that $\mathbb{D}x_{il}(t)=x_{il}(t+1)=x_{i(l+1)}(t)$ for $i\in N_0$ and $l=1,\dots,n-1$. Then, we have

$$\begin{aligned}&\lim_{t\to\infty}E[\big(x_{il}(t)-\bar{x}_l(t)\big)^2]\\=&\lim_{t\to\infty}E[\big(x_{in}(t-n+l)-\bar{x}_n(t-n+l)\big)^2]\\=&0,\end{aligned}$$

where $\bar{x}_l(t)$ is the l-th element of $\bar{x}(t)$. Thus,

$$\lim_{t\to\infty}E[\|x_i(t)-\bar{x}(t)\|^2]=0.$$

ii) Similarly, when $\beta>\frac{1}{2f_M}(\frac{\gamma^2\lambda_G^2}{\lambda_2^2(\gamma\lambda_2-1)}+\gamma\alpha+1)$ and $\gamma>\frac{1}{\lambda_2}$, by Theorem 1, we have

$$E[\|K_2x_i(t)-K_2\bar{x}(t)\|^2]=O(\frac{1}{t}).$$

Same as the proof of Part i), by Lemmas 4-5, there is $E[\big(x_{il}(t)-\bar{x}_l(t)\big)^2]=O(\frac{1}{t})$, $i\in N_0$, $l=1,\dots,n$. As a result, we know that

$$E[\|x_i(t)-\bar{x}(t)\|^2]=O\left(\frac{1}{t}\right).$$

This completes the proof. ■

V. Consensus of MASs under Markovian Switching Communication Networks

In practical networks, communication topologies often evolve with temporal correlations. Markovian switching naturally models such dynamics, capturing statistical dependencies and encompassing a diverse range of stochastic evolutions. This section extends the proposed one-bit consensus algorithm to the Markovian switching case and evaluates its performance.

Model the communication links between agents as undirected time-varying topology $G_{m(t)}=(N_0,E_{m(t)})$, whose dynamic is described by a homogeneous Markovian chain $\{m(t):t\in\mathbb{N}\}$ with a state space $\{1,2,\dots,h\}$, transition probability $p_{uv}=\mathbb{P}\{m(t)=v|m(t-1)=u\}$, and stationary distribution $\pi_u=\lim_{t\to\infty}\mathbb{P}\{m(t)=u\}$, for all $u,v\in\{1,2,\dots,h\}$. $N_0=\{1,\dots,N\}$ is the set of agents, and $E_{m(t)}\subseteq N_0\times N_0$ is the ordered edges set of the topology $G_{m(t)}$. Moreover, assume $G_{m(t)}\in\{G_1,G_2,\dots,G_h\}$. Denote $N_i^{m(t)}$ as the neighbor set of the agent i in the topology $G_{m(t)}$. Denote the adjacency matrix and the degree matrix at time t as $A_{m(t)}$ and $D_{m(t)}$, respectively. Then, the Laplace matrix of $G_{m(t)}$ is defined as $L_{m(t)}=D_{m(t)}-A_{m(t)}$.

To ensure the effectiveness of the algorithm, we give the following joint connectivity assumption.

Assumption 5: $\{G_1,G_2,\dots,G_h\}$ are jointly connected.

Based on Assumption 5, denote the jointly connected topology formed by G_1,G_2,…,G_h as $G'=(N_0,E')$, where $E'=E_1\cup\cdots\cup E_h$ is the set of all the edges. Next, same as the fixed topology G in Section IV, we define new notations for the new topology G', which is considered in the following. Without loss of generality, we continue to use the same notations d_i, N_i, d, $\hat{z}(t)$, $s(t)$, and C for G' to simplify this paper.

Similarly, three matrices are constructed for the switching topology case.

$P_{m(t)}$ is designed to select each neighbor of each agent at time t. Define $P_{m(t)}=\mathrm{diag}\{\dots,p_{m(t)}^{ij},\dots\}\in\mathbb{R}^{d\times d}$ in lexicographic order of $(i,j)\in E'$, where $p_{m(t)}^{ij}=1$ when $(i,j)\in E_{m(t)}$, else $p_{m(t)}^{ij}=0$.

$W_{m(t)}$ is designed to select the neighbor set of each agent at time t. Define $W_{m(t)}=[\dots,W_{m(t)}^{ij},\dots]\in\mathbb{R}^{N\times d}$ in lexicographic order of $(i,j)\in E'$, where $W_{m(t)}^{ij}=\vec{e}_i\in\mathbb{R}^N$ when $(i,j)\in E_{m(t)}$, else is $\vec{0}_N$.

Q is defined as Section IV and is based on the topology G'.

Based on the above matrices, the vector forms of estimation and update are given as follows:

1.Estimation:

$$\hat{z}(t)=\mathbf{\Pi}_{\boldsymbol{M}}\Big\{\hat{z}(t-1)+\frac{\beta}{t}P_{m(t)}\big(\mathcal{F}(C-\hat{z}(t-1))-s(t)\big)\Big\},$$

where $\mathbf{\Pi}_{\boldsymbol{M}}(z)=[\Pi_M(z_1),\dots,\Pi_M(z_d)]^T$, $\mathcal{F}(z)=[F(z_1),\dots,F(z_d)]^T$, for any $z=[z_1,z_2,\dots,z_d]^T\in\mathbb{R}^d$.

2.Update:

$$\begin{aligned}x(t)=&\Big(I_N\otimes(A+BK_1)-\frac{\gamma}{t}L_{m(t-1)}\otimes BK_2\Big)x(t-1)\\&+\frac{\gamma}{t}(W_{m(t-1)}\otimes B)\hat{\varepsilon}(t),\end{aligned}$$

where $\hat{\varepsilon}(t)=\hat{z}(t)-(Q\otimes K_2)x(t)$ is the estimation error of the compressed state $(I_N\otimes K_2)x(t)$.

The main difficulty in the analysis of the switching topology case is that the fixed matrices I_d, L, and W used to describe the relationship between agents are respectively changed into switching matrices $P_{m(t)}$, $L_{m(t)}$ and $W_{m(t)}$, in contrast to the fixed topology case. To overcome this challenge, the following lemmas are given to establish a certain equivalence relationship between the Markovian switching topologies and a fixed topology.

Lemma 6: For the switching matrices $L_{m(t)}$ and $P_{m(t)}$, we have the following conclusions,

$$E[L_{m(t)}] = \sum_{i=1}^{h} \pi_i L_i + O(\lambda_L^t),$$

$$E[P_{m(t)}] = \sum_{i=1}^{h} \pi_i P_i + O(\lambda_P^t),$$

where $0 < \lambda_L, \lambda_P < 1$.

Proof: Denote $p_{u,t} = \mathbb{P}\{G_{m(t)} = G_u\}$, the transition probability matrix as $P = \{p_{uv}\}_{u,v}$ and stationary distribution vector as $\pi = [\pi_1, \ldots, \pi_h]$. By the property of transition probability p_{uv} and stationary distribution π_u, we have $\pi P = \pi$, which implies that π is a positive left eigenvector corresponding to the eigenvalue 1.

It can be seen that $P\vec{1}_h = \vec{1}_h$, thus $\vec{1}_h$ is a positive right eigenvector corresponding to the eigenvalue 1. Besides, we can know that $\pi\vec{1}_h = \sum_{i=1}^{h} \pi_i = 1$.

Since P is symmetrical, by [39, Corollary 1], we know that eigenvalue 1 satisfies [39, Theorem 1.1]. Then, by [39, Theorem 1.2], there exists a λ_p that satisfies $0 < \lambda_p < 1$ such that

$$P^t = \vec{1}_h \pi + O(\lambda_p^t).$$

Moreover, by the definition of $p_{u,t}$, it can be seen that $[p_{1,t+1}, \ldots, p_{h,t+1}] = [p_{1,t}, \ldots, p_{h,t}]P$. Then, we have

$$\begin{aligned}[p_{1,t}, \ldots, p_{h,t}] =& [p_{1,1}, \ldots, p_{h,1}]P^{t-1} \\ =& [p_{1,1}, \ldots, p_{h,1}]\vec{1}_h \pi + O(\lambda_p^t),\end{aligned}$$

which implies that $p_{u,t} = \sum_{i=1}^{h} \pi_u p_{i,1} + O(\lambda_p^t) = \pi_u + O(\lambda_p^t)$, where $0 < \lambda_p < 1$.

Therefore, by the definition of expectation, we have $E[L_{m(t)}] = \sum_{i=1}^{h} p_{i,t} L_i = \sum_{i=1}^{h} \pi_i L_i + O(\lambda_L^t)$ and $E[P_{m(t)}] = \sum_{i=1}^{h} \pi_i P_i + O(\lambda_P^t)$, where $0 < \lambda_L, \lambda_P < 1$. This completes the proof. ■

Define $\check{L} = \sum_{i=1}^{h} \pi_i L_i$. The following lemma describes its properties, which are similar to those in the fixed topology case in Section II-B. To simplify the notation, we denote the eigenvalues of $\check{L}$ as $\lambda_1 \le \lambda_2 \le \lambda_3 \le \ldots \le \lambda_N$, consistent with the notation used in Section II-B.

Lemma 7: ([33, Lemma 2]). If Assumption 5 holds, then matrix $\check{L}$ has these properties:

i) $\check{L}$ is a nonnegative definite matrix with rank $n-1$ and eigenvalues $0 = \lambda_1 < \lambda_2 \le \lambda_3 \le \ldots \le \lambda_N$;
ii) There exists an orthogonal matrix T_G such that $T_G^{-1} \check{L} T_G = \text{diag}\{\lambda_1, \ldots, \lambda_N\}$.

Remark 9: The switching topology case can be converted into a fixed one by taking the expectation, requiring only minor adjustments in the analysis process to achieve convergence properties consistent with the fixed topology case. From Lemmas 6 and 7, we can see that the switching matrix $L_{m(t)}$ can be treated as the sum of the fixed matrix $\check{L}$ and the exponential term $O(\lambda_L^t)$ when taking a mathematical expectation. Since $O(\lambda_L^t)$ converges exponentially to zero, it is smaller than $O(\frac{1}{t^2})$, which does not affect the analysis of convergence.

Based on these lemmas, the coupling relationship between the two Lyapunov functions $V(t)$ and $R(t)$ is obtained, which is in the same form as in the fixed graph case, and only with a slight change in the definitions of the constant coefficients.

Lemma 8: Under Assumptions 1-2 and 5, $V(t)$ satisfies

$$V(t) \le \Big(1 - \frac{\gamma\lambda_2}{t}\Big)V(t-1) + \frac{\gamma\lambda_G}{t\lambda_2}R(t-1) + O\Big(\frac{1}{t^2}\Big),$$

where $\lambda_G = \max\limits_{1\le i\le h}\{\|T_G^{-1} J_N W_i\|^2\}$.

Proof: See Appendix III-A. ■

Lemma 9: Under Assumptions 1-2 and 5, $R(t)$ satisfies

$$\begin{aligned}R(t) \le& \Big(1 - \frac{2\beta f_M \pi_{\min} - \gamma\alpha}{t}\Big)R(t-1) + \frac{\gamma\lambda_G}{t\lambda_2}V(t-1) \\ &+ O\Big(\frac{1}{t^2}\Big),\end{aligned}$$

where $f_M = \min_{i,j\in N_0}\{f(|c_{ij}| + M)\}$, $\pi_{\min} = \min\limits_{1\le i\le h}\{\pi_i\}$, $\alpha = 2\sqrt{\lambda_{QW}} + \lambda_{QL}\lambda_2/\lambda_G$, $\lambda_{QW} = \max\limits_{1\le i\le h}\{\|QW_i\|^2\}$, and $\lambda_{QL} = \max\limits_{1\le i\le h}\{\|QL_i T_G\|^2\}$.

Proof: See Appendix III-B. ■

Since Lemmas 8-9 have the same form as Lemmas 2-3, we can just repeat the analysis process of the fixed topology case to get the results of the switching topology case, as the following theorems.

Theorem 3: Under Assumptions 1-2 and 5, the compressed states $K_2 x_j(t)$ and their estimates $\hat{z}_{ij}(t)$ satisfy:

i) When $\beta > \frac{1}{2 f_M \pi_{\min}}(\frac{\gamma\lambda_G^2}{\lambda_2^3} + \gamma\alpha)$, the compressed states reach consensus and their estimates converge to the real compressed states, i.e., for $i \in N_0$, $j \in N_i$,

$$\lim_{t\to\infty} E[\|K_2 x_i(t) - K_2\bar{x}(t)\|^2] = 0,$$

$$\lim_{t\to\infty} E[\|\hat{z}_{ij}(t) - K_2 x_j(t)\|^2] = 0;$$

ii) When $\beta > \frac{1}{2 f_M \pi_{\min}}(\frac{\gamma^2\lambda_G^2}{\lambda_2^2(\gamma\lambda_2 - 1)} + \gamma\alpha + 1)$ and $\gamma > \frac{1}{\lambda_2}$, the compressed states reach consensus at the rate of $O(\frac{1}{t})$, and their estimates converge to the real compressed states at the rate of $O(\frac{1}{t})$, i.e., for $i \in N_0$, $j \in N_i$,

$$E[\|K_2 x_i(t) - K_2\bar{x}(t)\|^2] = O\Big(\frac{1}{t}\Big),$$

$$E[\|\hat{z}_{ij}(t) - K_2 x_j(t)\|^2] = O\Big(\frac{1}{t}\Big),$$

where $\bar{x}(t) = \frac{1}{N}\sum_{i=1}^{N} x_i(t)$.

Theorem 4: Under Assumptions 1-2 and 4-5, the following results are obtained:

i) The MAS (1)-(3) reaches consensus, i.e.,

$$\lim_{t\to\infty} E[\|x_i(t) - \bar{x}(t)\|^2] = 0,$$

providing with $\beta > \frac{1}{2 f_M \pi_{\min}}(\frac{\gamma\lambda_G^2}{\lambda_2^3} + \gamma\alpha)$;

ii) The MAS (1)-(3) reaches consensus at the rate of $O(\frac{1}{t})$, i.e.,

$$E[\|x_i(t) - \bar{x}(t)\|^2] = O\left(\frac{1}{t}\right),$$

if $\beta > \frac{1}{2f_M\pi_{\min}}(\frac{\gamma^2\lambda_G^2}{\lambda_2^2(\gamma\lambda_2-1)} + \gamma\alpha + 1)$ and $\gamma > \frac{1}{\lambda_2}$.

The difference between the results of the Markovian switching topologies and the fixed topology is that the step coefficient β depends on the switching probability of Markovian switching topologies, i.e., the minimal stationary distribution $\pi_{\min}$ of the Markovian chain $\{m(t) : t \in \mathbb{N}\}$. In the case of a single topology, it is straightforward to observe that $\pi_{\min} = 1$. Consequently, Theorems 3-4 reduce to the results for the fixed topology case, as stated in Theorems 1-2.

VI. Numerical Simulation

In this section, we provide two simulation examples to illustrate the theoretical results.

Example 1: Consider the altitude consensus control of a multi-aircraft system composed of seven aircraft ([20] and [40]), whose communication network is shown as Figure 1.

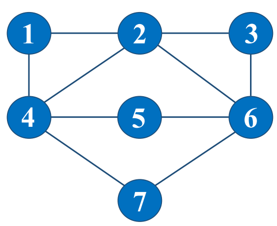

Fig. 1. The fixed communication topology of the multi-agent system

The schematic diagram of the aircraft is shown in Figure 2, where L_W denotes the lift force applied at the center of lift C_L; C_G is the center of mass; d is the distance between C_L and C_G. The mass of aircraft is denoted by m and its moment of inertia about C_G is denoted by J. The altitude of the ith aircraft is denoted by h_i, which is controlled by the elevator's rotation E_i.

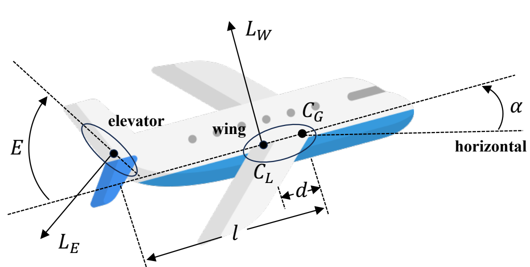

Fig. 2. A schematic diagram of an aircraft

By [40], the dynamics of the ith aircraft can be modelled as follows:

$$\begin{cases} J\ddot{\alpha}_i + b\dot{\alpha}_i + (C_{ZE}l + C_{ZW}d)\dot{\alpha}_i = C_{ZE}lE_i, \\ m\ddot{h}_i = (C_{ZE} + C_{ZW})\alpha_i - C_{ZE}E_i, \end{cases} \quad (9)$$

where α_i is the rotation angle of the ith aircraft about C_G; b is the friction coefficient; $L_E = C_{ZE}(E_i - \alpha_i)$ is the aerodynamic force on the elevator.

In this numerical simulation, similar to [20], we set $J = 1, m = 1, b = 4, C_{ZE} = 1, C_{ZW} = 5, l = 3$ and $d = 0.2$. Let $x_i = (\alpha_i, \dot{\alpha}_i, h_i, \dot{h}_i)^T$. Then, the dynamics of the ith aircraft can be rewritten as

$$\dot{x}_i = A_c x_i + B_c E_i,$$

where

$$A_c = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -4 & -4 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 6 & 0 & 0 & 0 \end{bmatrix}, B_c = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 1 \end{bmatrix}.$$

Due to the wide application of digital networks, the data is sampled in practice. In the simulation, the sampling period is set to be $T = 0.5$. Then, by adopting the zero-order holder strategy, the discretization of (9) is obtained as follows:

$$x_i(t+1) = A_d x_i(t) + B_d E_i(t),$$

where

$$A_d = e^{A_c T} = \begin{bmatrix} 0.7358 & 0.1839 & 0 & 0 \\ -0.7358 & 0 & 0 & 0 \\ 0.7073 & 0.0777 & 1 & 0.5 \\ 2.6891 & 0.3964 & 0 & 1 \end{bmatrix},$$

$$B_d = \int_0^T e^{A_c t} dt B_c = \begin{bmatrix} 0.1982 \\ 0.5518 \\ -0.093 \\ -0.2668 \end{bmatrix}. \quad (10)$$

It is easily verified that the linear system described by (10) is controllable. Therefore, it can be transformed into the Brunovsky canonical form defined by (3).

In this simulation example, we set the communication noises $d_{ij} \sim N(0, 16)$ for $i \in N_0$, $j \in N_i$. Then, assume that the initial altitudes h_i of these seven aircraft are $h_1 = 5, h_2 = 2, h_3 = 4, h_4 = 3, h_5 = 1.5, h_6 = 2.5, h_7 = 1$ and the initial values of $\alpha_i, \dot{\alpha}_i, \dot{h}_i$ are 0. The unit of h_i can be selected as kilometers, meters, etc., according to the actual situation. Denote the initial estimates as $\vec{0}_{20}$. Moreover, select $\beta = 1500$ and $\gamma = 1$. Then, we apply the consensus algorithm in Algorithm 1, with the thresholds $c_{ij} = -2$, the controller gain $K_1 = [-0.9224, -0.1825, -0.0000, -0.1788]$ and the compression coefficient $K_2 = [3.8734, 0.9054, 0.3575, 0.8772]$. Given the initial states x_i^0 and compression coefficient K_2, choose $M = 2$ as appropriate.

As shown in Figure 3, the altitudes of all agents reach consensus. Besides, Figure 4 shows a linear relationship between the logarithm of the mean square errors (MSEs) and the logarithm of the index t, which illustrates each agent can reach consensus at the rate of $O(\frac{1}{t})$.

We next construct another simulation example to demonstrate the consensus of the MAS under Markovian switching communication networks.

Example 2: Take the same system settings, including the system model, noise distribution, and initial states, as those in Example 1. The Markovian switching topologies $G_{m(t)}$

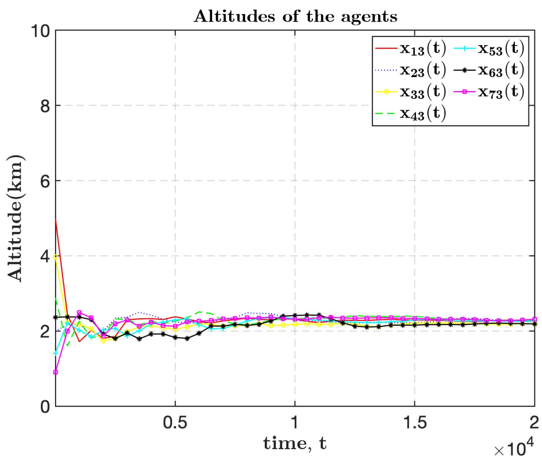

Fig. 3. The altitude trajectory of each aircraft under fixed topology

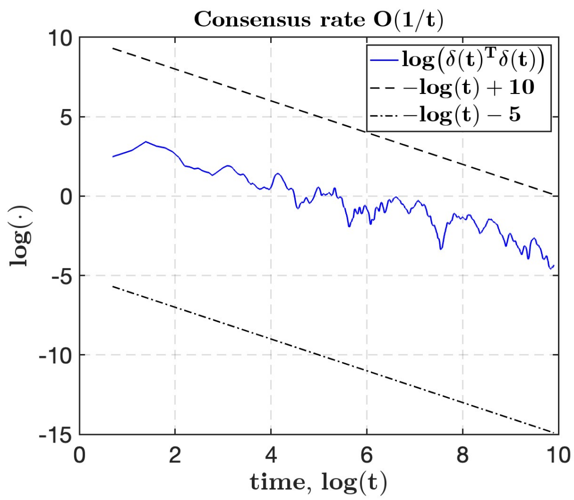

Fig. 4. The trajectory of the log(MSEs) under fixed topology

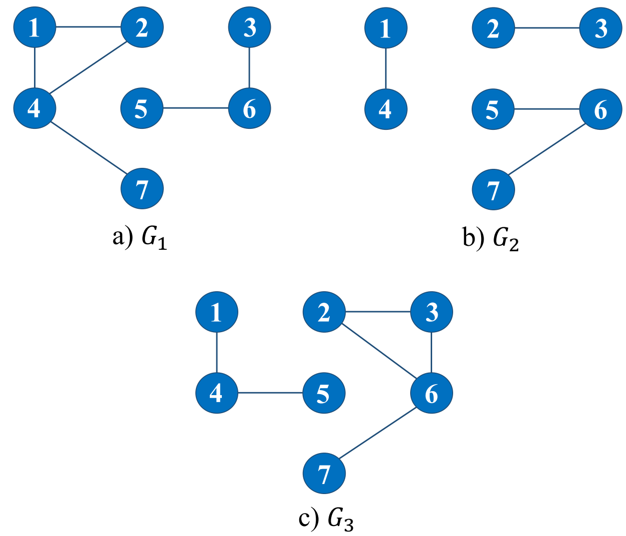

Fig. 5. The Markovian switching communication topologies

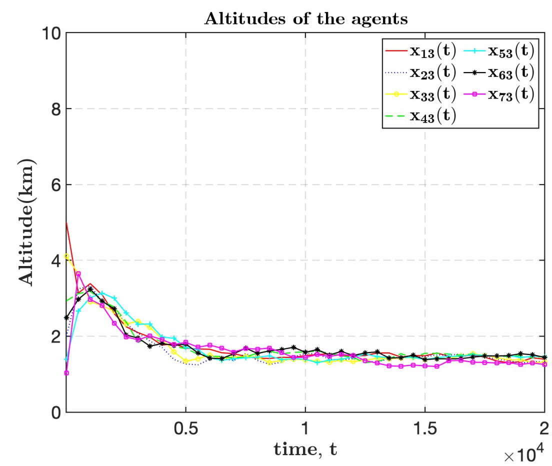

Fig. 6. The altitude trajectory of each aircraft under Markovian switching topologies

with $m(t) \in \{1, 2, 3\}$ are depicted in Figure 5. The transition probability matrix of the Markov chain $m(t)$ is chosen to be $P = \begin{bmatrix} 0.5 & 0.3 & 0.2 \\ 0.2 & 0.5 & 0.3 \\ 0.3 & 0.2 & 0.5 \end{bmatrix}$ with its stationary distribution is $\pi = [1/3, 1/3, 1/3]$. It is clear that Assumption 5 is satisfied.

By Theorem 4, we select $\beta = 10000$, and $\gamma = 2.4$. Then, apply the consensus algorithm with the same thresholds c_{ij}, the controller gain K_1, and the compression coefficient K_2 as Example 1. Figure 6 shows the consensus of the MAS under Markovian switching topologies. Moreover, Figure 7 illustrates the consensus rate $O(\frac{1}{t})$ of the MAS, which is the same as Example 1.

Remark 10: The connectivity and switching probability of communication topologies affect the selection of the step coefficients β and γ. Specifically, the values of β and γ are affected by the algebraic connectivity λ_2 of the topology graph and minimum stationary distribution probability $\pi_{\min}$ related to the switching probability. To achieve consensus, smaller values of λ_2 and $\pi_{\min}$ require larger step coefficients, as illustrated in Examples 1-2. To be specific, in the Markovian switching topology case, both the decrease of λ_2 and $\pi_{\min}$ increase the lower bound on the required estimation step coefficient β and control step coefficient γ, necessitating a larger estimation step coefficient β. This explains why the estimation step coefficient β is selected as $\beta = 1500$ in Example 1 and $\beta = 10000$ in Example 2.

VII. Conclusion

This paper investigates the one-bit consensus of controllable linear MASs with communication noises. A consensus algorithm consists of a communication protocol and a consensus controller is designed. The communication protocol introduces a linear compression encoding function to achieve one-bit communication, which significantly saves communication costs. A consensus controller with both a stabilization term and a consensus term is proposed to ensure consensus of an unstable MAS. Two Lyapunov functions for the consensus error and estimation error of the compressed states are constructed. By jointly analyzing the convergence property of them, it is shown that the compressed states can achieve

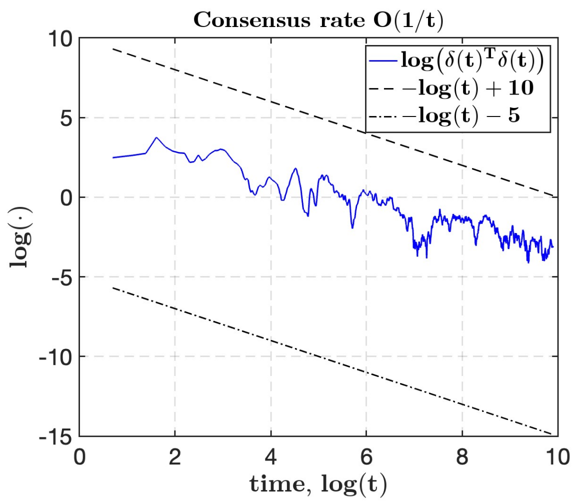

Fig. 7. The trajectory of the log(MSEs) under Markovian switching topologies

consensus, and the estimates can converge to the real values. Moreover, by establishing the consensus equivalence between the compressed and original states, it is proven that all agents can achieve consensus at the rate of the reciprocal of the iteration times, both in the case with fixed and switching communication networks.

In the future, there will be many interesting problems in one-bit consensus control. For example, the one-bit consensus with event-triggered communication mechanisms has garnered considerable attention, yet numerous open questions and challenges remain in this area that warrant further investigation.

Appendix I
The Proof About Compressed States and Estimates

A. The proof of Lemma 2

Step 1: Calculate the recursive expression of $\delta(t)$.

From $\delta(t)=(J_N\otimes I_n)x(t)$, $L\vec{1}_N=\vec{1}_N^T L=0$, and $J_N L=LJ_N$, we have

$$\begin{aligned}\delta(t)=&(J_N\otimes I_n)x(t)\\=&(J_N\otimes I_n)(I_n\otimes\tilde{A}-\frac{\gamma}{t}L\otimes BK_2)x(t-1)\\&+\frac{\gamma}{t}(J_NW\otimes B)\hat{\varepsilon}(t-1)\\=&(I_N\otimes\tilde{A}-\frac{\gamma}{t}L\otimes BK_2)\delta(t-1)\\&+\frac{\gamma}{t}(J_NW\otimes B)\hat{\varepsilon}(t-1),\end{aligned}\tag{A1}$$

where $\tilde{A}=A+BK_1$, $\delta(t)=[\delta_1^T(t),\dots,\delta_N^T(t)]^T\in\mathbb{R}^{nN}$, and $\delta_i(t)=x_i(t)-\bar{x}(t)$ for $i\in N_0$.

Step 2: Transform $\delta(t)$ to $\hat{\delta}(t)$ and show that its first element is zero.

Denote $\tilde{\delta}(t)=(T_G^{-1}\otimes I_n)\delta(t)$. Denote the first n elements of $\tilde{\delta}(t)$ by $\tilde{\delta}^{(1)}(t)$, and the others by $\tilde{\delta}^{(2)}(t)$. Since the first row of T_G^{-1} is $\frac{1}{\sqrt{N}}\vec{1}_N^T$, we know $\tilde{\delta}^{(1)}(t)=\frac{1}{\sqrt{N}}\sum_{i=1}^N\delta_i(t)=\vec{0}_n$.

Denote $\hat{\delta}(t)=(I_N\otimes K_2)\tilde{\delta}(t)$. Denote the first element of $\hat{\delta}(t)$ by $\hat{\delta}^{(1)}(t)$, and the others by $\hat{\delta}^{(2)}(t)$. Since $\hat{\delta}^{(1)}(t)=K_2\tilde{\delta}^{(1)}(t)$ and $\tilde{\delta}^{(1)}(t)=\vec{0}_n$, we know $\hat{\delta}^{(1)}(t)=0$. Besides, it can be seen that $V(t)=E[\|\hat{\delta}(t)\|^2]=E[\hat{\delta}^T(t)\hat{\delta}(t)]=E[\hat{\delta}^{(2)T}(t)\hat{\delta}^{(2)}(t)]$.

Step 3: Expand the Lyapunov function $V(t)$ in terms of $\hat{\delta}(t)$ and $\hat{\varepsilon}(t)$.

Then, by Remark 5 and (A1), we have

$$\begin{aligned}V(t)=&E[\|(T_G^{-1}\otimes K_2)\delta(t)\|^2]\\=&E\Big[\Big\|(I_N-\frac{\gamma}{t}T_G^{-1}LT_G)\hat{\delta}(t-1)\\&+\frac{\gamma}{t}(T_G^{-1}J_NW)\hat{\varepsilon}(t-1)\Big\|^2\Big]\\=&E[\hat{\delta}^T(t-1)(I_N-\frac{\gamma}{t}T_G^{-1}LT_G)^2\hat{\delta}(t-1)]\\&+\frac{2\gamma}{t}E[\hat{\delta}^T(t-1)(I_N-\frac{\gamma}{t}T_G^{-1}LT_G)\\&\cdot(T_G^{-1}J_NW)\hat{\varepsilon}(t-1)]+O\Big(\frac{1}{t^2}\Big).\end{aligned}\tag{A2}$$

Denote the first and second items of formula (A2) as $V_1(t)$ and $V_2(t)$, respectively, i.e., $V(t)\triangleq V_1(t)+V_2(t)+O\big(\frac{1}{t^2}\big)$.

Step 4: Estimate $V_1(t)$ and express it in terms of $V(t-1)$ to derive a recursive inequality of $V(t)$.

By $\hat{\delta}^{(1)}(t)=0$, we have $\hat{\delta}^T(t)(I_N-\frac{\gamma}{t}\mathrm{diag}(0,\lambda_2,\dots,\lambda_N))\cdot\hat{\delta}(t)=\hat{\delta}^{(2)T}(t)(I_{N-1}-\frac{\gamma}{t}\mathrm{diag}(\lambda_2,\dots,\lambda_N))\hat{\delta}^{(2)}(t)$. Therefore, one can get

$$\begin{aligned}V_1(t)=&E[\hat{\delta}^T(t-1)(I_N-\frac{\gamma}{t}T_G^{-1}LT_G)^2\hat{\delta}(t-1)]\\=&E\Big[\hat{\delta}^T(t-1)\mathrm{diag}^2(1,1-\frac{\gamma}{t}\lambda_2\dots,1-\frac{\gamma}{t}\lambda_N)\hat{\delta}(t-1)\Big]\\=&E\Big[\hat{\delta}^{(2)T}(t-1)\mathrm{diag}^2(1-\frac{\gamma}{t}\lambda_2\dots,1-\frac{\gamma}{t}\lambda_N)\hat{\delta}^{(2)}(t-1)\Big]\\\le&(1-\frac{\gamma\lambda_2}{t})^2V(t-1).\end{aligned}\tag{A3}$$

Step 5: Estimate $V_2(t)$ and express it in terms of $V(t-1)$ and $R(t-1)$.

From the property of $T_G^{-1}LT_G$ and Cauchy Schwarz inequality, we have

$$\begin{aligned}V_2(t)=&\frac{2\gamma}{t}E[\hat{\delta}^T(t-1)(I_N-\frac{\gamma}{t}T_G^{-1}LT_G)\\&\cdot(T_G^{-1}J_NW)\hat{\varepsilon}(t-1)]\\=&\frac{2\gamma}{t}E[\hat{\delta}^T(t-1)\mathrm{diag}(1,1-\frac{\gamma}{t}\lambda_2\dots,1-\frac{\gamma}{t}\lambda_N)\\&\cdot(T_G^{-1}J_NW)\hat{\varepsilon}(t-1)]\\\le&\frac{2\gamma}{t}\Big(E\big[\hat{\delta}^T(t-1)\mathrm{diag}^2(1,1-\frac{\gamma}{t}\lambda_2,\dots,1-\frac{\gamma}{t}\lambda_N)\\&\cdot\hat{\delta}(t-1)\big]E\big[\hat{\varepsilon}^T(t-1)W^TJ_N^TJ_NW\hat{\varepsilon}(t-1)\big]\Big)^{\frac{1}{2}}\\=&\frac{2\gamma}{t}\Big(E\big[\hat{\delta}^{(2)T}(t-1)\mathrm{diag}^2(1-\frac{\gamma}{t}\lambda_2,\dots,1-\frac{\gamma}{t}\lambda_N)\\&\cdot\hat{\delta}^{(2)}(t-1)\big]E\big[\hat{\varepsilon}^T(t-1)W^TJ_N^TJ_NW\hat{\varepsilon}(t-1)\big]\Big)^{\frac{1}{2}}\\\le&\frac{2\gamma}{t}\Big((1-\frac{\gamma\lambda_2}{t})^2V(t-1)\Big)^{\frac{1}{2}}\Big(\lambda_GR(t-1)\Big)^{\frac{1}{2}}\\\le&\frac{\gamma}{t}\Big(\lambda_2(1-\frac{\gamma\lambda_2}{t})^2V(t-1)+\frac{\lambda_G}{\lambda_2}R(t-1)\Big).\end{aligned}\tag{A4}$$

Step 6: Combine the inequalities of $V_1(t)$ and $V_2(t)$ to obtain the recursive inequality of $V(t)$.

Combining (A2)-(A4), we have

$$V(t) \leq \big(1-\frac{\gamma\lambda_2}{t}-\frac{\gamma^2\lambda_2^2}{t^2}+\frac{\gamma^3\lambda_2^3}{t^3}\big)V(t-1)+\frac{\gamma\lambda_G}{\lambda_2 t}R(t-1) + O\Big(\frac{1}{t^2}\Big)$$
$$\leq \big(1-\frac{\gamma\lambda_2}{t}\big)V(t-1)+\frac{\gamma\lambda_G/\lambda_2}{t}R(t-1)+O\Big(\frac{1}{t^2}\Big).$$

B. The proof of Lemma 3

Step 1: Express the Lyapunov function $R(t)$ in terms of $\hat{\varepsilon}(t)$ and $\hat{\delta}(t)$.

By the definition of $\hat{\varepsilon}(t)$, Lemma 1 and the nonexpansiveness of the projection mapping, we have

$$\begin{aligned}
R(t) =&E[\|\hat{\varepsilon}(t)\|^2]\\
=&E[\|\hat{z}(t)-(Q\otimes K_2)x(t)\|^2]\\
=&E\Big[\Big\|\mathbf{\Pi}_{\boldsymbol{M}}\Big\{\hat{z}(t-1)+\frac{\beta}{t}\big(\mathcal{F}(C-\hat{z}(t-1))-s(t)\big)\Big\}\\
&-(Q\otimes K_2)x(t)\Big\|^2\Big]\\
\leq&E\Big[\Big\|\hat{z}(t-1)+\frac{\beta}{t}\big(\mathcal{F}(C-\hat{z}(t-1))-s(t)\big)\\
&-(Q\otimes K_2)x(t)\Big\|^2\Big]\\
=&E\Big[\Big\|\big(I_d-\frac{\gamma}{t}QW\big)\hat{\varepsilon}(t-1)+\frac{\gamma}{t}(QL\otimes K_2)\delta(t-1)\\
&+\frac{\beta}{t}\big(\mathcal{F}(C-\hat{z}(t-1))-s(t)\big)\Big\|^2\Big]\\
=&E\Big[\Big\|\big(I_d-\frac{\gamma}{t}QW\big)\hat{\varepsilon}(t-1)+\frac{\gamma}{t}QLT_G\hat{\delta}(t-1)\\
&+\frac{\beta}{t}\big(\mathcal{F}(C-\hat{z}(t-1))-s(t)\big)\Big\|^2\Big]\\
=&E\Big[\hat{\varepsilon}^T(t-1)\big(I_d-\frac{\gamma}{t}QW\big)^T\big(I_d-\frac{\gamma}{t}QW\big)\hat{\varepsilon}(t-1)\Big]\\
&+\frac{2\gamma}{t}E\Big[\hat{\varepsilon}^T(t-1)\big(I_d-\frac{\gamma}{t}QW\big)^TQLT_G\hat{\delta}(t-1)\Big]\\
&+\frac{2\beta}{t}E\Big[\hat{\varepsilon}^T(t-1)\big(I_d-\frac{\gamma}{t}QW\big)^T\Big(\mathcal{F}\big(C-\hat{z}(t-1)\big)\\
&-s(t)\Big)\Big]+O\Big(\frac{1}{t^2}\Big).
\end{aligned}\tag{A5}$$

Denote the first, second, and third items of (A5) as $R_1(t)$, $R_2(t)$, and $R_3(t)$, respectively, i.e., $R(t)\leq R_1(t)+R_2(t)+R_3(t)+O(\frac{1}{t^2})$.

Step 2: Estimate $R_1(t)$ and express it in terms of $R(t-1)$.

$$\begin{aligned}
R_1(t) =&E\Big[\hat{\varepsilon}^T(t-1)\big(I_d-\frac{\gamma}{t}QW\big)^T\big(I_d-\frac{\gamma}{t}QW\big)\hat{\varepsilon}(t-1)\Big]\\
\leq&\big(1+\frac{\gamma\sqrt{\lambda_{QW}}}{t}\big)^2R(t-1).
\end{aligned}\tag{A6}$$

Step 3: Estimate $R_2(t)$ and also express it in terms of $V(t-1)$ and $R(t-1)$.

Based on the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
R_2(t) =&\frac{2\gamma}{t}E\Big[\hat{\varepsilon}^T(t-1)\big(I_d-\frac{\gamma}{t}QW\big)^TQLT_G\hat{\delta}(t-1)\Big]\\
\leq&\frac{2\gamma}{t}\Big(E\Big[\hat{\varepsilon}^T(t-1)\big(I_d-\frac{\gamma}{t}QW\big)^T\big(I_d-\frac{\gamma}{t}QW\big)\\
&\cdot\hat{\varepsilon}(t-1)\Big]E[\hat{\delta}^T(t-1)T_G^TL^TQ^TQLT_G\hat{\delta}(t-1)]\Big)^{\frac{1}{2}}\\
\leq&\frac{2\gamma}{t}\Big(\big(1+\frac{\gamma\sqrt{\lambda_{QW}}}{t}\big)^2R(t-1)\cdot\lambda_{QL}V(t-1)\Big)^{\frac{1}{2}}\\
\leq&\frac{\gamma}{t}\Big(\frac{\lambda_{QL}\lambda_2}{\lambda_G}\big(1+\frac{\gamma\sqrt{\lambda_{QW}}}{t}\big)^2R(t-1)+\frac{\lambda_G}{\lambda_2}V(t-1)\Big).
\end{aligned}\tag{A7}$$

Step 4: Express $R_3(t)$ in terms of $V(t-1)$ and $R(t-1)$.

By the communication protocol designed in Algorithm 1, it can be seen that $E[s(t)]=\mathcal{F}\big(C-(Q\otimes K_2)x(t)\big)$ under Assumption 2. Then, we get

$$\begin{aligned}
R_3(t) =&\frac{2\beta}{t}E\Big[\hat{\varepsilon}^T(t-1)\big(I_d-\frac{\gamma}{t}QW\big)^T\Big(\mathcal{F}\big(C-\hat{z}(t-1)\big)\\
&-s(t)\Big)\Big]\\
=&\frac{2\beta}{t}E\Big[\hat{\varepsilon}^T(t-1)\big(I_d-\frac{\gamma}{t}QW\big)^T\Big(\mathcal{F}\big(C-\hat{z}(t-1)\big)\\
&-\mathcal{F}\big(C-(Q\otimes K_2)x(t)\big)\Big)\Big]
\end{aligned}$$

And, by Lagrange's Mean Value Theorem, we have

$$\begin{aligned}
&F\big(c_{ij}-\hat{z}_{ij}(t-1)\big)-F\big(c_{ij}-K_2x_j(t)\big)\\
=&-f(\zeta_{ij}(t))\big(\hat{z}_{ij}(t-1)-K_2x_j(t)\big),
\end{aligned}$$

where $\zeta_{ij}(t)$ is between $c_{ij}-\hat{z}_{ij}(t-1)$ and $c_{ij}-K_2x_j(t)$.

Let $\zeta(t)=[\dots,\zeta_{ij}(t),\dots]^T$, arranged in lexicographic order of $(i,j)\in E$. Denote $\mathrm{diag}(\vec{f}(\zeta(t)))=\mathrm{diag}\{\dots,f(\zeta_{ij}(t)),\dots\}\in\mathbb{R}^{d\times d}$ as a diagonal matrix generated by each element of the vector $\vec{f}(\zeta(t))=[\dots,f(\zeta_{ij}(t)),\dots]^T\in\mathbb{R}^d$. By Lemma 1, $\zeta_{ij}(t)$ is bounded. Since the function $f(\cdot)$ is continuous, we have $\mathrm{diag}\big(\vec{f}(\zeta(t))\big)\geq f_M\cdot I_d$ and

$$\begin{aligned}
R_3(t) =&-\frac{2\beta}{t}E\Big[\hat{\varepsilon}^T(t-1)\big(I_d-\frac{\gamma}{t}QW\big)^T\mathrm{diag}\big(\vec{f}(\zeta(t))\big)\\
&\cdot\big(\hat{z}(t-1)-(Q\otimes K_2)x(t)\big)\\
=&-\frac{2\beta}{t}E\Big[\hat{\varepsilon}^T(t-1)\big(I_d-\frac{\gamma}{t}QW\big)^T\mathrm{diag}\big(\vec{f}(\zeta(t))\big)\\
&\cdot\Big(\hat{\varepsilon}(t-1)-\frac{\gamma}{t}\big(QW\hat{\varepsilon}(t-1)-QLT_G\hat{\delta}(t-1)\big)\Big)\Big]\\
=&-\frac{2\beta}{t}E\big[\hat{\varepsilon}^T(t-1)\mathrm{diag}\big(\vec{f}(\zeta(t))\big)\hat{\varepsilon}(t-1)\big]+O\Big(\frac{1}{t^2}\Big)\\
\leq&-\frac{2\beta f_M}{t}R(t-1)+O\Big(\frac{1}{t^2}\Big).
\end{aligned}\tag{A8}$$

Step 5: Combine the inequalities of $R_1(t)$, $R_2(t)$, and $R_3(t)$ to obtain the recursive inequality of $R(t)$.

Considering (A5) with (A6)-(A8), we can obtain that

$$R(t)\leq\Big(1-\frac{2\beta f_M-\gamma\alpha}{t}\Big)R(t-1)+\frac{\gamma\lambda_G/\lambda_2}{t}V(t-1) +O\Big(\frac{1}{t^2}\Big).$$

Appendix II
The proof about original states

A. The proof of Lemma 4

By the Brunovsky canonical form (3) and (6), we have

$$x_i(t+1)=(A+BK_1)x_i(t)+\frac{\gamma}{t+1}B\sum_{j\in N_i}(\hat{z}_{ij}(t)-K_2x_j(t))$$

$$=\begin{bmatrix}0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 1\\ b_1 & b_2-b_1 & \cdots & 1-b_{n-1}\end{bmatrix}x_i(t)+\frac{\gamma}{t+1}\begin{bmatrix}0\\ \vdots\\ 0\\ 1\end{bmatrix}\sum_{j\in N_i}(\hat{z}_{ij}(t)-K_2x_j(t)).$$

Then, there is $x_{i1}(t+1)=x_{i2}(t),\ldots,x_{i(n-1)}(t+1)=x_{in}(t)$. From $\mathbb{D}x_{ij}(t)=x_{ij}(t+1)$, we have

$$\mathbb{D}x_{ij}(t)=x_{i(j+1)}(t),\quad j=1,\ldots,n-1.$$

Since $K_2x_i(t)=b_1x_{i1}(t)+\ldots,+b_{n-1}x_{i(n-1)}(t)+x_{in}(t)$, we can obatin that

$$\mathbb{D}^{n-1}K_2x_i(t)=b_1x_{in}(t)+\ldots+b_{n-1}\mathbb{D}^{n-2}x_{in}(t)+\mathbb{D}^{n-1}x_{in}(t).$$

B. The proof of Lemma 5

i) **Step 1:** Prove the convergence of $\prod_{i=2}^{n-1}(\mathbb{D}-r_i)\xi(t)$.

Let $\xi_1(t)\triangleq\prod_{i=2}^{n-1}(\mathbb{D}-r_i)\xi(t)$. Then, $\mathbb{D}\xi_1(t)=r_1\xi(t)+\eta(t)$, i.e., $\xi_1(t+1)=r_1\xi_1(t)+\eta(t)$, and thus,

$$\begin{aligned}\xi_1(t)&=r_1^t\xi_1(0)+\sum_{i=0}^{t-1}r_1^i\eta(t-1-i)\\ &=r_1^t\xi_1(0)+\sum_{i=0}^{t-1}r_1^i\big(\eta(t-1-i)-\eta^*\big)+\sum_{i=0}^{t-1}r_1^i\eta^*,\end{aligned}\tag{B1}$$

where

$$\begin{aligned}&E\Big[\Big(\sum_{i=0}^{t-1}r_1^i\big(\eta(t-1-i)-\eta^*\big)\Big)^2\Big]\\ =&E\Big[\sum_{i=0}^{t-1}r_1^{2i}\big(\eta(t-1-i)-\eta^*\big)^2+\sum_{i=0}^{t-1}\sum_{j\neq i}r_1^{i+j}\big(\eta(t-1-i)-\eta^*\big)\big(\eta(t-1-j)-\eta^*\big)\Big]\\ =&\sum_{i=0}^{t-1}r_1^{2i}E\big[\big(\eta(t-1-i)-\eta^*\big)^2\big]+\sum_{i=0}^{t-1}\sum_{j\neq i}r_1^{i+j}E\big[\big(\eta(t-1-i)-\eta^*\big)\big(\eta(t-1-j)-\eta^*\big)\big]\\ \leq&\sum_{i=0}^{t-1}r_1^{2i}E\big[\big(\eta(t-1-i)-\eta^*\big)^2\big]+\sum_{i=0}^{t-1}\sum_{j\neq i}r_1^{i+j}\Big(E\big[\big(\eta(t-1-i)-\eta^*\big)^2\big]E\big[\big(\eta(t-1-j)-\eta^*\big)^2\big]\Big)^{\frac{1}{2}}\\ =&\Big(\sum_{i=0}^{t-1}r_1^i\Big(E\big[\big(\eta(t-1-i)-\eta^*\big)^2\big]\Big)^{\frac{1}{2}}\Big)^2.\end{aligned}\tag{B2}$$

To calculate the above formula (B2), without loss of generality, assume that $r_1\geq 0$. Then,

$$\begin{aligned}&\lim_{t\to\infty}\Big|\sum_{i=0}^{t-1}r_1^i\Big(E\big[\big(\eta(t-1-i)-\eta^*\big)^2\big]\Big)^{\frac{1}{2}}\Big|\\ =&\lim_{t\to\infty}\frac{\sum_{i=0}^{t-1}r_1^{-i}\Big(E\big[\big(\eta(i)-\eta^*\big)^2\big]\Big)^{\frac{1}{2}}}{r_1^{1-t}}.\end{aligned}\tag{B3}$$

Since $0\leq r_1<1$, we know that r_1^{1-t} is a strictly monotone and divergent sequence. Then, by Stolz-Cesàro theorem,

$$\begin{aligned}&\lim_{t\to\infty}\frac{\sum_{i=0}^{t-1}r_1^{-i}\Big(E\big[\big(\eta(i)-\eta^*\big)^2\big]\Big)^{\frac{1}{2}}}{r_1^{1-t}}\\ =&\lim_{t\to\infty}\frac{r_1^{-t}\Big(E\big[\big(\eta(t)-\eta^*\big)^2\big]\Big)^{\frac{1}{2}}}{r_1^{-t}-r_1^{1-t}}\\ =&\lim_{t\to\infty}\frac{\Big(E\big[\big(\eta(t)-\eta^*\big)^2\big]\Big)^{\frac{1}{2}}}{1-r_1}\\ =&0.\end{aligned}$$

Therefore, by (B2) we have

$$\lim_{t\to\infty}E\Big[\Big(\sum_{i=0}^{t-1}r_1^i\big(\eta(t-1-i)-\eta^*\big)\Big)^2\Big]=0.$$

Since $|r_1|<1$ and (B1), we have

$$\begin{aligned}&\lim_{t\to\infty}E\big[\big(\xi_1(t)-\frac{1}{1-r_1}\eta^*\big)^2\big]\\ =&\lim_{t\to\infty}E\Big[\Big(r_1^t\xi_1(0)+\sum_{i=0}^{t-1}r_1^i\big(\eta(t-1-i)-\eta^*\big)+\sum_{i=0}^{t-1}r_1^i\eta^*-\frac{1}{1-r_1}\eta^*\Big)^2\Big]\\ =&\lim_{t\to\infty}E\Big[\Big(r_1^t\xi_1(0)+\sum_{i=0}^{t-1}r_1^i\big(\eta(t-1-i)-\eta^*\big)-\frac{r_1^t\eta^*}{1-r_1}\Big)^2\Big]\\ =&\lim_{t\to\infty}r_1^{2t}\xi_1^2(0)+2\lim_{t\to\infty}r_1^t\xi_1(0)E\big[\sum_{i=0}^{t-1}r_1^i\big(\eta(t-1-i)-\eta^*\big)\big]\\ &+\lim_{t\to\infty}E\Big[\Big(\sum_{i=0}^{t-1}r_1^i\big(\eta(t-1-i)-\eta^*\big)\Big)^2\Big]\\ &-2\lim_{t\to\infty}\frac{r_1^t\eta^*}{1-r_1}E\big[\sum_{i=0}^{t-1}r_1^i\big(\eta(t-1-i)-\eta^*\big)\big]\\ &-2\lim_{t\to\infty}\frac{r_1^{2t}\eta^*\xi_1(0)}{1-r_1}+\lim_{t\to\infty}(\frac{r_1^t\eta^*}{1-r_1})^2\\ =&0.\end{aligned}$$

Step 2: Obtain the convergence of $\xi(t)$.

Thus, $\lim_{t\to\infty}E\big[\big(\xi_1(t)-\frac{1}{1-r_1}\eta^*\big)^2\big]=0$. Similarly, denoting $\xi_i(t)\triangleq\prod_{j=i+1}^{n-1}(\mathbb{D}-r_j)\xi(t)$, repeating the procedure, we have $\lim_{t\to\infty}E\big[\big(\xi_i(t)-\frac{1}{\prod_{j=1}^{i}1-r_j}\eta^*\big)^2\big]=0$, for $i=1,\ldots,n-1$. By the definition of $\xi_i(t)$, we know that $\xi(t)=\xi_{n-1}(t)$. Thus,

$$\lim_{t\to\infty}E\big[\big(\xi(t)-\xi^*\big)^2\big]=0,$$

where $\xi^*=\frac{1}{\prod_{j=1}^{n-1}1-r_j}\eta^*$.

ii) **Step 1:** Calculate the convergence rate of $\prod_{i=2}^{n-1}(\mathbb{D}-r_i)\xi(t)$.

By Part i), we know that $\xi_i(t)$ converges to $\frac{1}{\prod_{j=1}^{i}1-r_j}\eta^*$ in the mean square when $\eta(t)$ converges to η^* in the mean square. Now we calculate the convergence rate of $\xi_i(t)$ under the condition that $\eta(t)$ converges to η^* at the rate of $O(\frac{1}{t})$.

Firstly, we calculate $\left|\sum_{i=0}^{t-1} r_1^i \left(E\left[\left(\eta(t-1-i)-\eta^*\right)^2\right]\right)^{\frac{1}{2}}\right|$. Without loss of generality, assume that $r_1 \geq 0$. Then, (B3) is obtained.

Since $E[(\eta(t)-\eta^*)^2] = O(\frac{1}{t})$, there exists $M_\eta > 0$ such that $E[(\eta(t)-\eta^*)^2] \leq \frac{M_\eta}{t}$. Then,

$$\begin{aligned} &\frac{\sum_{i=0}^{t-1} r_1^{-i}\left(E\left[\left(\eta(i)-\eta^*\right)^2\right]\right)^{\frac{1}{2}}}{r_1^{1-t}} \\ \leq &\frac{\sum_{i=0}^{t-1} r_1^{-i} M_\eta \sqrt{i}}{r_1^{1-t}}. \end{aligned} \tag{B4}$$

For sequence $r_1^{1-t}/\sqrt{t}$, it can be seen that $r_1^{1-t}/\sqrt{t}$ is strictly monotone and divergent when $t \geq -1/\ln r_1$. Then, by Stolz-Cesàro theorem,

$$\begin{aligned} &\lim_{t\to\infty} \frac{\sum_{i=0}^{t-1} r_1^{-i} M_\eta \sqrt{i}}{r_1^{1-t}/\sqrt{t}} \\ =&\lim_{t\to\infty} \frac{r_1^{-t} M_\eta/\sqrt{t}}{r_1^{-t}/\sqrt{t+1} - r_1^{1-t}/\sqrt{t}} \\ =&\frac{\sqrt{M_\eta}}{1-r_1}, \end{aligned}$$

which implies $\frac{\sum_{i=0}^{t-1} r_1^{-i} M_\eta \sqrt{i}}{r_1^{1-t}/\sqrt{t}} = O(1)$, or equivalently, $\frac{\sum_{i=0}^{t-1} r_1^{-i} M_\eta \sqrt{i}}{r_1^{1-t}} = O(\frac{1}{\sqrt{t}})$. This together with (B2)-(B4) gives

$$E\Big[\Big(\sum_{i=0}^{t-1} r_1^i \big(\eta(t-1-i)-\eta^*\big)\Big)^2\Big] = O\Big(\frac{1}{t}\Big).$$

Thus, $E\big[\big|\sum_{i=0}^{t-1} r_1^i\big(\eta(t-1-i)-\eta^*\big)\big|\big] = O(1/\sqrt{t})$ can also be obtained. By (B1), we can get that

$$\begin{aligned} &E\big[\big(\xi_1(t) - \frac{1}{1-r_1}\eta^*\big)^2\big] \\ =&r_1^{2t}\xi_1^2(0) + 2r_1^t\xi_1(0)E\big[\sum_{i=0}^{t-1} r_1^i\big(\eta(t-1-i)-\eta^*\big)\big] \\ &+ E\Big[\Big(\sum_{i=0}^{t-1} r_1^i\big(\eta(t-1-i)-\eta^*\big)\Big)^2\Big] - 2\frac{r_1^{2t}\eta^*\xi_1(0)}{1-r_1} \\ &- 2\frac{r_1^t\eta^*}{1-r_1}E\big[\sum_{i=0}^{t-1} r_1^i\big(\eta(t-1-i)-\eta^*\big)\big] + \big(\frac{r_1^t\eta^*}{1-r_1}\big)^2 \\ =&O(r_1^{2t}) + O\Big(\frac{r_1^t}{\sqrt{t}}\Big) + O\Big(\frac{1}{t}\Big) \\ =&O\Big(\frac{1}{t}\Big). \end{aligned}$$

Step 2: Obtain the convergence rate of $\xi(t)$.

Similarly to the proof of Part i), repeating the procedure, we have

$$E\big[\big(\xi(t)-\xi^*\big)^2\big] = O\Big(\frac{1}{t}\Big).$$

Appendix III
The proof in the switching topology case

A. The proof of Lemma 8

Repeating the analysis process in Appendix I-A, we can conclude that

$$\begin{aligned} \delta(t) =&(I_N \otimes \tilde{A} - \frac{\gamma}{t}L_{m(t-1)} \otimes BK_2)\delta(t-1) \\ &+ \frac{\gamma}{t}(J_N W_{m(t-1)} \otimes B)\hat{\varepsilon}(t-1), \end{aligned} \tag{C1}$$

and

$$\begin{aligned} V(t) =&E[\hat{\delta}^T(t-1)(I_N - \frac{\gamma}{t}T_G^{-1}L_{m(t-1)}T_G)^2\hat{\delta}(t-1)] \\ &+ \frac{2\gamma}{t}E[\hat{\delta}^T(t-1)(I_N - \frac{\gamma}{t}T_G^{-1}L_{m(t-1)}T_G) \\ &\cdot (T_G^{-1}J_N W_{m(t-1)})\hat{\varepsilon}(t-1)] + O\Big(\frac{1}{t^2}\Big). \end{aligned} \tag{C2}$$

It can be seen that the only differences between (A1) and (C1), (A2) and (C2) are that the fixed matrices L and W have been modified into switching matrices $L_{m(t)}$ and $W_{m(t)}$.

Then, as Remark 9 says, by the property of conditional expectation and Lemma 6, we can get that $E[\hat{\delta}^T(t)L_{m(t)}\hat{\delta}(t)] = E[E[\hat{\delta}^T(t)L_{m(t)}\hat{\delta}(t)|\hat{\delta}(t)]] = E[\hat{\delta}^T(t)\check{L}\hat{\delta}(t)] + O(\lambda_L^t)$, thus dealing with the switching matrix $L_{m(t)}$ in the following. To be specific, we have

$$\begin{aligned} V_1(t) =&E[\hat{\delta}^T(t-1)(I_N - \frac{\gamma}{t}T_G^{-1}L_{m(t-1)}T_G)^2\hat{\delta}(t-1)] \\ =&E[\hat{\delta}^T(t-1)(I_N - \frac{2\gamma}{t}T_G^{-1}L_{m(t-1)}T_G)\hat{\delta}(t-1)] \\ &+ O\Big(\frac{1}{t^2}\Big) \\ =&E\big[\hat{\delta}^T(t-1)\big(I_N - \frac{2\gamma}{t}T_G^{-1}\big(\check{L} + O(\lambda_L^t)\big)T_G\big)\hat{\delta}(t-1)\big] \\ &+ O\Big(\frac{1}{t^2}\Big) \\ =&E\big[\hat{\delta}^T(t-1)\big(I_N - \frac{2\gamma}{t}T_G^{-1}\check{L}T_G\big)\hat{\delta}(t-1)\big] + O\Big(\frac{1}{t^2}\Big), \end{aligned}$$

which is similar to the form of $V_1(t)$ in Appendix I-A.

At this point, the switching matrix $L_{m(t)}$ has been transformed into the fixed matrix $\check{L}$. By repeating the proof procedure from Lemma 2 in Appendix I-A, we can derive the following results.

$$V_1(t) \leq \big(1 - \frac{2\gamma\lambda_2}{t}\big)V(t-1) + O\Big(\frac{1}{t^2}\Big), \tag{C3}$$

$$V_2(t) \leq \frac{\gamma}{t}\Big(\lambda_2\big(1 - \frac{2\gamma\lambda_2}{t}\big)V(t-1) + \frac{\lambda_G}{\lambda_2}R(t-1)\Big) + O\big(\frac{1}{t^2}\big), \tag{C4}$$

where the definition of $\lambda_G = \max_{1\leq i\leq h}\{\|T_G^{-1}J_N W_i\|^2\}$ is different with the fixed topology case and λ_2 is the minimum non-negative eigenvalue of the union topology G'.

Combining (C2)-(C4), we have

$$V(t) \leq \big(1 - \frac{\gamma\lambda_2}{t}\big)V(t-1) + \frac{\gamma\lambda_G/\lambda_2}{t}R(t-1) + O\Big(\frac{1}{t^2}\Big),$$

which is consistent with the fixed topology case.

B. The proof of Lemma 9

Changing the fixed matrices L, W into $L_{m(t)}, W_{m(t)}$ and $P_{m(t)}$ and repeating the analysis process in Appendix I-B, we can conclude that

$$\begin{aligned}R(t) =&E[\|\hat{\varepsilon}(t)\|^2]\\ =&E\Big[\Big\|\mathbf{\Pi}_{\boldsymbol{M}}\Big\{\hat{z}(t-1)+\frac{\beta}{t}P_{m(t)}\big(\mathcal{F}(C-\hat{z}(t-1))\\ &-s(t)\big)\Big\}-(Q\otimes K_2)x(t)\Big\|^2\Big]\\ =&E\Big[\hat{\varepsilon}^T(t-1)\big(I_d-\frac{\gamma}{t}QW_{m(t-1)}\big)^T\big(I_d-\frac{\gamma}{t}QW_{m(t-1)}\big)\\ &\cdot\hat{\varepsilon}(t-1)\Big]+\frac{2\gamma}{t}E\Big[\hat{\varepsilon}^T(t-1)\big(I_d-\frac{\gamma}{t}QW_{m(t-1)}\big)^TQ\\ &\cdot L_{m(t-1)}T_G\hat{\delta}(t-1)\Big]+\frac{2\beta}{t}E\Big[\hat{\varepsilon}^T(t-1)\big(I_d-\frac{\gamma}{t}Q\\ &\cdot W_{m(t-1)}\big)^TP_{m(t)}\Big(\mathcal{F}\big(C-\hat{z}(t-1)\big)-s(t)\big)\Big)\Big]\\ &+O\Big(\frac{1}{t^2}\Big). \end{aligned} \tag{C5}$$

Denote the first, second, and third items of (C5) as $R_1(t)$, $R_2(t)$, and $R_3(t)$, respectively, i.e., $R(t)\le R_1(t)+R_2(t)+R_3(t)+O\big(\frac{1}{t^2}\big)$. Then, we have

$$R_1(t)\le\big(1+\frac{\gamma\sqrt{\lambda_{QW}}}{t}\big)^2R(t-1), \tag{C6}$$

where $\lambda_{QW}=\max\limits_{1\le i\le h}\{\|QW_i\|^2\}$ is different with the fixed topology case.

Similarly to (A7), we can get that

$$R_2(t)\le\frac{\gamma}{t}\Big(\frac{\lambda_{QL}\lambda_2}{\lambda_G}\big(1+\frac{\gamma\sqrt{\lambda_{QW}}}{t}\big)^2R(t-1)+\frac{\lambda_G}{\lambda_2}V(t-1)\Big), \tag{C7}$$

where $\lambda_{QL}=\max\limits_{1\le i\le h}\{\|QL_iT_G\|^2\}$ and λ_2 are different with (A7).

Subsequently, using the conclusion in Appendix I-B, since $\sum_{i=1}^h\pi_iP_i\ge\pi_{\min}I_N$, it can be seen that

$$\begin{aligned}R_3(t)=&-\frac{2\beta}{t}E\big[\hat{\varepsilon}^T(t-1)P_{m(t)}\mathrm{diag}\big(\vec{f}(\zeta(t))\big)\hat{\varepsilon}(t-1)\big]\\ &+O\Big(\frac{1}{t^2}\Big)\\ =&-\frac{2\beta}{t}E\big[\hat{\varepsilon}^T(t-1)(\sum_{i=1}^h\pi_iP_i)\mathrm{diag}\big(\vec{f}(\zeta(t))\big)\hat{\varepsilon}(t-1)\big]\\ &+O\Big(\frac{1}{t^2}\Big)\\ \le&-\frac{2\beta f_M\pi_{\min}}{t}R(t-1)+O\Big(\frac{1}{t^2}\Big). \end{aligned} \tag{C8}$$

Considering (C5) with (C6)-(C8), we can obtain that

$$\begin{aligned}R(t)\le&\Big(1-\frac{2\beta f_M\pi_{\min}-\gamma\alpha}{t}\Big)R(t-1)+\frac{\gamma\lambda_G/\lambda_2}{t}V(t-1)\\ &+O\Big(\frac{1}{t^2}\Big),\end{aligned}$$

which has a new constant $\pi_{\min}$ that corresponding with the switching topologies.

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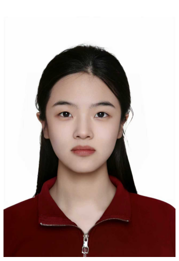

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